

**PRINCIPLE RESULTS OF INTRODUCTORY REAL ANALYSIS**

Term Paper for Dr. Hooper's Real Analysis Course

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## ABSTRACT

Many books on real analysis contain extensive material on the real number system. This includes many results regarding sequences and series of real numbers as well as results from metric topology as they relate to the real number system. Royden is brief in this respect and I will omit it entirely since I feel it is introductory and should be prerequisite of a real analysis course. Also, I will omit any mention of most of the practical calculus techniques (particularly differential calculus) which are generally taught in introductory calculus courses. However, introductory real analysis does not extend beyond calculus of one variable, so multivariable calculus is not prerequisite of this essay.

I will begin by defining the Riemann integral and discuss some of the theorems relating to it. Throughout this section most of the sets mentioned will be intervals on the real number line. In the next section I will discuss "measurable" sets (still on the real number line) and functions. In the third section I will define the Lebesgue integral and discuss some of the results of Lebesgue integration. Finally, I will discuss how Lebesgue integration extends Riemann integration, that is, why we are not content with an integral defined only in terms of intervals, which are more tractable than

measurable sets. Proofs will be omitted for the sake of brevity and ease of exposition.

Since this wordprocessor does not have an epsilon symbol, I will use alpha in a similar manner.  $\bar{E}$  represents the complement of the set  $E$ . The symbol  $<$  means either "is less than" or "is a subset of." The context should make it clear which is meant.

## Section 1

## Riemann Integration

The term "almost everywhere" appears throughout real analysis. Almost everywhere in a set is everywhere except on a subset of measure zero. "Measure" will be defined in section 2 but it is possible (and useful) to define "measure zero" separately. A set  $E$  is of measure zero (denoted  $\mu E = 0$ ) if, for all  $\alpha > 0$ , there exists a countable sequence of open intervals  $I_n$  which cover  $E$  and  $\sum |I_n| < \alpha$ . If  $I$  is an interval then  $|I|$  represents its length. Clearly, a countable union of sets, each of measure zero, is also of measure zero. Also, any countable set is of measure zero. This is remarkable considering that there is a countable set (the rational numbers) which is dense in the reals. The reals are of measure  $\infty$ . It may seem that all sets of measure zero would be countable. However, the Cantor set is uncountably infinite and of measure zero. This is also remarkable because, in probability, measure zero means (roughly) "impossible", though there may be uncountably many counterexamples.

Several concepts must be defined in preparation for defining the Riemann integral. Unless explicitly stated otherwise, all intervals and functions will be assumed to be

bounded.  $M[f;I] = \sup f(x)$  and  $m[f;I] = \inf f(x)$  with  $x$  in the interval  $I$ .  $\sigma = \langle x_k \rangle$  with  $i = 0, 1, \dots, n$  is a subdivision of  $[a,b]$  if  $a = x_0 < x_1 < \dots < x_n = b$ . To simplify the notation, all integrals in this section, unless otherwise noted, will be over  $[a,b]$ . Also,  $\sigma$  and  $\tau$  will always be subdivisions of this interval and  $I_k$  the interval  $[x_{k-1}, x_k]$ . With regards to a particular subdivision  $\sigma$ , the upper and lower sums of  $f$  in  $[a,b]$  are  $U[f;\sigma] = \sum M[f;I_k] \cdot |I_k|$  and  $L[f;\sigma] = \sum m[f;I_k] \cdot |I_k|$ , respectively. Clearly, we have  $L[f;\sigma] \leq U[f;\tau]$ . Finally, the upper and lower Riemann integrals are defined as  $\sup U[f;\sigma]$  and  $\inf L[f;\sigma]$  over all subdivisions of  $[a,b]$ . From the previous result we have the lower Riemann integral less than or equal to the upper Riemann integral. If they are exactly equal then  $f$  is Riemann integrable over  $[a,b]$  and this common value is its Riemann integral, denoted

$$\int f(x) dx$$

The  $R$  will be dropped as will will consider no other kind of integral in this section. In section 3 the Lebesgue integral will be defined and, from there on, only it will be considered so there will be no need to denote it with an  $L$ .

It follows directly from the definition that  $c \in (a,b)$  implies that the Riemann integrals over  $[a,c]$  and  $[c,b]$  exist and sum to the Riemann integral over  $[a,b]$ . Also, by defining the integral over  $[a,b]$  to be the negative of the integral over  $[b,a]$ , the statement above is true regardless of the order of  $a$ ,  $b$ , and  $c$ .

$$c \int f(x)dx + \int g(x)dx = \int cf(x) + g(x)dx$$

with the integrability of the function on the left implying that of the function on the right.

For  $f(x) \leq g(x)$  almost everywhere,

$$\int f(x)dx \leq \int g(x)dx \quad \text{and} \quad \left| \int f(x)dx \right| \leq \int |f(x)|dx$$

Roughly, a subdivision cannot be made fine enough that any of its component intervals are contained in the set of points where  $g(x) < f(x)$ .

This is also why the characteristic function of any dense subset of measure zero (or its complement) is not Riemann integrable. A subdivision cannot be made fine enough that any of its component intervals are contained in the defining set. This is clear from the definition of zero measure. For instance, the characteristic function of rational numbers in  $[a,b]$  is not Riemann integrable. It may seem that the difficulty is only that its points of discontinuity are dense in  $[a,b]$ . However, consider the function defined on  $[0,1]$  such that, for all rational numbers  $x = n/d$  (in lowest terms),  $f(x) = 1/d$  and, for all irrational numbers,  $f(x) = 0$ . Because the irrationals are dense in  $[0,1]$ , there is one in every neighborhood of a rational number so, taking  $\alpha = 1/(2d)$  for each rational number, they are points of discontinuity and are dense in  $[0,1]$ . Clearly, the lower Riemann integral of  $f(x)$  is zero. Let  $\sigma_n$  be the subdivision with  $x_k = k/n$  and  $U[f;\sigma_n] = 1/n$ , that is,  $U[f;\sigma_n]$  is the area of  $n$  squares, each  $1/n$  on a side.  $\sup U[f;\sigma_n]$  for all  $n$  is 0, so the Riemann integral of  $f$  is defined and is zero. Thus, density

of discontinuities alone is not sufficient. As the example in the next paragraph shows, discontinuities of measure zero alone is not sufficient.

$f$  is Riemann integrable iff, for all  $\alpha > 0$ , there exists a subdivision  $\sigma$  such that  $U[f;\sigma] < L[f;\sigma] + \alpha$ . This is clear from the definition but it is not useful in practice because it would require finding such a subdivision. It is a fundamental result of integration theory that  $f$  is Riemann integrable iff  $f$  is almost everywhere continuous. Consider the characteristic function of the Cantor set. The Cantor set is the set of numbers in  $[0,1]$  with a ternary expansion that does not contain the digit 1. At every point in the Cantor set, this function is discontinuous. This is because, for any larger element, there is a number not in the Cantor set between them obtained by replacing the first 2 of the larger element with a 1. At every point of  $[0,1]$  not in the Cantor set, this function is continuous because, if its first 1 is in the  $n$ 'th place, all the numbers equal to it in their first  $n$  places are not in the Cantor set and form an open interval  $1/3^n$  wide. For this same reason, the Cantor set is not dense. For reasons given in section 2, the Cantor set is of measure zero and, hence, Riemann integrable. For the same reason that binary expansions are uncountably infinite, the Cantor set is also uncountably infinite. Thus, it has as many discontinuities as a function can have without losing Riemann integrability.

The derivative of  $f(x)$  is  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Considering that  $f$  is Riemann integrable iff  $f$  is almost

everywhere continuous, one might ask if this is true of derivatives as well. Differentiability implies continuity but the converse is not true. The absolute value function is continuous but is not differentiable at zero. Periodic triangular waveforms are everywhere continuous but are not differentiable on a countably infinite set. By taking a countable summation of such functions it is possible to construct an everywhere continuous but nowhere differentiable function. Let  $f_1(x) = |x|$  in  $[-\frac{1}{2}, \frac{1}{2}]$  and  $f_1(x+n) = f_1(x)$  for all real  $x$  and integer  $n$ . For  $n = 2, 3, \dots$ , let  $f_n(x) = f_1(4^{n-1}x)/4^{n-1}$ .  $\sum f_n(x)$  converges by the Weierstrass M-test because  $|f_n(x)| \leq 1/4^{n-1}$ . Let  $h_n = \pm 1/4^{n+1}$  so we have  $|f_n(x+h_n) - f_n(x)| = |h_n|$ . Also,  $|f_m(x+h_n) - f_m(x)| = |h_n|$  for  $m \leq n$  but zero for  $m > n$ . Thus,  $(f(x+h_n) - f(x))/h_n$  is an even or odd integer exactly where  $n$  is even or odd and, consequently, the derivative of  $f(x)$  does not exist.

For a function to have a derivative almost everywhere on  $[a,b]$ , it must be, not just continuous, but absolutely continuous. That is, given  $\alpha > 0$ , there exists  $\delta > 0$  such that  $\sum |x_i' - x_i| < \delta$  implies  $\sum |f(x_i') - f(x_i)| < \alpha$  with  $\{(x_i' - x_i)\}$  a finite collection of non-overlapping intervals in  $[a,b]$ . Absolute continuity is equivalent to bounded variation, which is easier to verify. Bounded variation on  $[a,b]$  means that  $\sup \sum |f(x_k) - f(x_{k-1})| < \infty$ . Other than direct recourse to the definition of differentiability, there is no way to upgrade almost everywhere to everywhere. This can be a problem in practical applications such as Newton's Method, which assumes the existence of derivatives.

If a function is differentiable, will its derivative necessarily be continuous? No. Consider  $f(0) = 0$  and  $f(x) = x^2 \sin(1/x)$  elsewhere. It is differentiable but its derivative is discontinuous at 0. If a function is continuous it is necessarily the derivative of some function, however. That is, if  $f$  is continuous at  $x$  then there exists a function  $F$  such that  $F'(x) = f(x)$ . Furthermore, not only can we prove the existence of such a function but we can give it a name.

$$F(x) = \int_a^x f(t) dt \quad \text{with } x \in [a, b]$$

Notice that, while  $F(x)$  exists, it is not unique. It depends on one's choice of  $a$ . Furthermore,  $F$  may not be expressible in terms of the standard transcendental functions. For instance,  $f(x) = \exp(-\frac{1}{2}x^2)/\sqrt{2\pi}$  is everywhere continuous and hence has an anti-derivative  $F(x) = \frac{1}{2}\text{erf}(x/\sqrt{2})$ , but there is no expression for the error function,  $F(x)$ , other than as the indefinite integral of  $f(x)$ .

Thus, the derivative is related to the integral. This is remarkable considering that their definitions make no mention of one another and do not look much alike. In fact, it is so remarkable that the theorem above is called the Fundamental Theorem of Calculus.

There is another (more useful) version of the Fundamental Theorem of Calculus which requires two preliminary theorems. If  $f'(x) = 0$  on  $[a, b]$  then  $f(x) = c$  on  $[a, b]$  and, immediately, if  $f'(x) = g'(x)$  on  $[a, b]$ , then  $f(x) - g(x) = c$  on  $[a, b]$ . And so we have our most important result: If  $f$  is

continuous on  $[a,b]$  and  $F$  an anti-derivative of it, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Differentiation is easy. Integration is harder but, having associated derivatives with integrals, we can use the chain rule to prove the very useful method of substitution. If  $g'(x)$  is continuous on  $[a,b]$  and  $f(x)$  is continuous on  $g([a,b])$ , then

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx$$

Integration by parts is derived from the product rule.

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Recall that all intervals and functions have been assumed to be bounded. If we relax this requirement for intervals we get improper integrals of the first kind. The theory of such integrals closely parallels that of sequences and series. Partial integrals play the part of partial sums and absolute convergence implies convergence but not the converse. For non-negative, non-increasing functions the improper integral converges iff the infinite series does. The concept of rearranging the terms in a series does not carry over to integration, however, because subdivisions only have a finite number of terms. Thus,

$$\int_{\pi}^{\infty} \frac{\sin(x)}{x} dx = \int_{\pi}^{\infty} \frac{\cos(x)}{x^2} dx + \frac{1}{\pi}$$

even though the integral on the left is conditionally

convergent and the one on the right is absolutely convergent. This could not happen for an infinite series because, with appropriate rearrangement of its terms, a conditionally convergent series can be made to converge to any real number.

If we retain the requirement that intervals be bounded but relax the requirement that functions be bounded, we have improper integrals of the second kind. These are called Cauchy-Riemann integrals. If the function is unbounded on a (small) finite set of points in  $[a,b]$ , one can partition  $[a,b]$  at those points (called singularities) and evaluate only over intervals with singularities at their boundary. If the singularity is on the left then one takes a right-hand limit

$$\lim_{\alpha \rightarrow 0^+} \int_{a+\alpha}^b f(x) dx$$

Similarly with left-hand limits for singularities on the right. These limits are handled in much the same way that improper integrals of the first kind are. In fact they can often be converted into one another as

$$\lim_{\alpha \rightarrow 0^+} \int_{\alpha}^1 \frac{1}{x} dx = \lim_{\alpha \rightarrow 0^+} \int_1^{1/\alpha} \frac{1}{u} du$$

with  $u = 1/x$ . If there are a lot (or an infinity) of singularities this process is more difficult.

Improper integrals of both the first and second kind can be partitioned so one first evaluates on the side with the singularity and then does the tail. Thus, it is never necessary to deal with both improprieties at once, though

both integrals must be finite for the combined integral to be finite.

## Section 2

## Lebesgue Measure

In section 1 virtually all the sets considered were intervals and, in particular, all the integrals were evaluated over bounded intervals. Even for improper integrals of the first kind all the integrals considered were evaluated over bounded intervals. One just took a sequence of such integrals. In fact, for most applications involving real numbers one only considers intervals. However, particularly in probability, it is sometimes necessary to consider (and measure) more complicated sets. The measure of an interval is its length, so all that has been said in section 1 regarding intervals still holds in the more general setting of Lebesgue measure. We will not go beyond the real numbers, however, so "set" will always refer to a subset of the real number line.

The outer measure of a set is  $m^* = \inf \sum |I_n|$  with  $\langle I_n \rangle$  any countable collection of intervals which cover the set. Clearly, the outer measure of the union of a countable collection of sets is less than or equal to the sum of the outer measures of all the sets, because they may overlap. Thus,  $m^*A \leq m^*(A \cap E) + m^*(A \cap \sim E)$ . If, for all A, we also have  $m^*A \geq m^*(A \cap E) + m^*(A \cap \sim E)$ , then E is measurable and

$\mu E = m^*E$ . This definition is due to Carathéodory.

Measurability is not much of a restriction. Royden, and every other author that I consulted, give the same highly artificial example of a set that is not measurable. It may be that there is no other example, or at least none that can be constructed without using the Axiom of Choice to pick from an uncountably infinite collection of sets, which is a somewhat dubious procedure. Thus, for practical applications, one can assume that all sets are measurable. In particular, those of measure zero considered in section 1 as well as unbounded intervals and all open and closed sets are measurable.

This is not the only way to define Lebesgue measure. Because of how the general version of the Lebesgue integral is defined, we really only need to consider sets of finite measure. Thus, one may begin by taking  $[a,b]$  rather than the entire real line as the universal set. This does not change the definition of outer measure, which can be abbreviated as  $m^*E = \inf |G|$  with  $G$  an open covering of  $E$ . Now we can define inner measure as  $m_*E = \sup |F| = \sup (b - a - |\sim F|)$ , however, with  $F$  a closed set contained in  $E$ . Measurable sets are exactly those whose inner and outer measure are equal and this common value is their Lebesgue measure. With this in mind, all sets mentioned from hereon will be of finite measure unless explicitly stated otherwise.

This definition leads directly to the Littlewood's First Principle, that the measurability of  $E$  implies the existence of a countable union of closed sets  $F$  and a countable

intersection of open sets  $G$  such that  $F < E < G$  and  $E \setminus F$  and  $G \setminus E$  are both of measure zero. If "countable" is replaced with "finite" then we only have  $\mu(E \setminus F) < \alpha$  and  $\mu(G \setminus E) < \alpha$  for arbitrarily small  $\alpha$ . Because defining both inner and outer measure leads so easily to Littlewood's First Principle, I prefer this definition rather than Carathéodory's definition (cited by Royden) which uses only outer measure.

Carathéodory's definition requires proving that a relation holds for every possible set  $A$ , which does not seem very useful. There is even another definition which also involves only outer measure, that is,  $E$  is measurable iff, given  $\alpha > 0$ , there exists open sets  $G_1$  and  $G_2$  such that  $E < G_1$ ,  $\bar{E} < G_2$ , and  $|G_1 \cap G_2| < \alpha$ .

Actually, it does not really matter how measureability is defined since, for all practical purposes, all sets are measurable. With this in mind, most of the theorems about measurable sets can be skimmed over as they establish only existence. Countable additivity of disjoint sets is important because it can be used to find the measure of sets. In particular, for nested sets  $E_1 < E_2 < \dots$ , the measure of their union is the limit of their measures. Similarly, for nested sets  $E_1 > E_2 > \dots$ , the measure of their intersection is the limit of their measures. This latter result can be used to show that the Cantor set is of measure zero. I suspect that most sets whose measure is actually known are the union or the intersection of nested sets. Also, having established the measure of a set, if its symmetric difference with another set is of measure zero, then that set has the