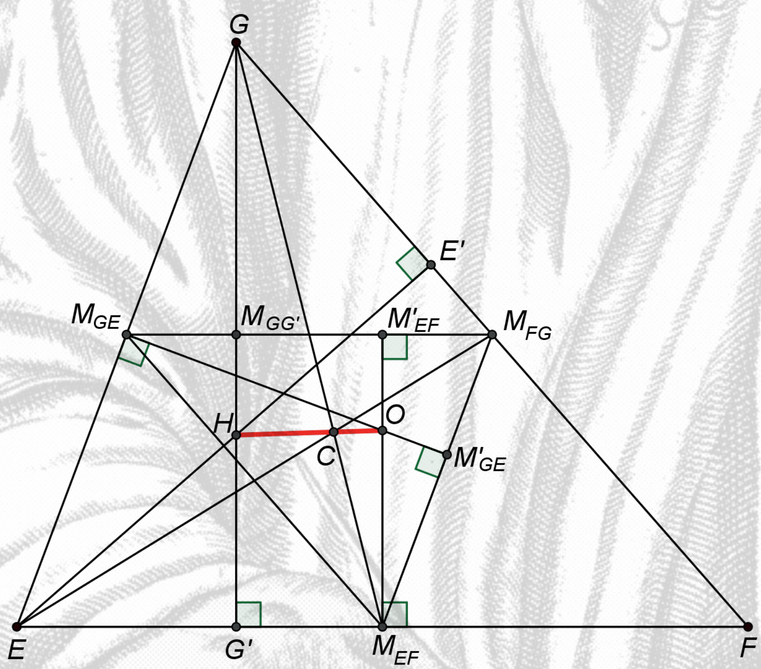
Geometry–Do

Volume One: Geometry without Multiplication

by Victor Aguilar, author of *Axiomatic Theory of Economics*

Postulates

1. Two points fully define the segment between them.
2. By extending it, a segment fully defines a line.
3. Three noncollinear points fully define a triangle.
4. The center and the radius fully define a circle.
5. All right angles are equal; equivalently, all straight angles are equal.
6. A line and a point not on it fully define the parallel through that point.



This book was **NOT** funded by the Bill and Melinda Gates Foundation.

A journey of discovery? A death march? It's the same thing!

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Volume One: Geometry without Multiplication

www.researchgate.net/profile/Victor_Aguilar4

The authorized PDF files can be downloaded from my Research Gate profile. When I first began this project in 2016, some early drafts of this book were posted at Research Gate and elsewhere for the purpose of eliciting comments and suggestions from mathematicians. These early drafts are much shorter than the completed 428-page book and contain many mistakes that have since been corrected. Use **ONLY** the PDF files currently posted on Research Gate – check before citing!

I only discuss geometry with mathematicians and engineers. I do not reply to American high-school teachers because they are just education majors who, while filled with fancy education theories, have never studied the content of the subject they currently happen to teach. If you do not have a degree granted by a mathematics or an engineering department, do not contact me. I have no respect for the EdD degree, I will not call you "doctor," and I will not speak to you.

This book was **NOT** funded by the Bill and Melinda Gates Foundation. I owe them nothing.

Acknowledgements

I want to thank Milan Zlatanović and Victor Tomno, who found an inordinate number of mistakes, all of which I fixed. I offered $100 to anyone who found a mistake in my manuscript. Most mathematicians gave up after finding one or none but, for a few months, Victor Tomno was making more money than I do. I had to deplete my savings! I paid for them all though, judging from the expression on my face, you would have thought I had to pull every one of those hundred-dollar bills out of my nose.

Also, my Russian translator Katerina Odnorozhenko found many mistakes, which greatly improved both the English and the Russian editions. Ogunkola Christiana found quite a few mistakes, and there are many more who found at least one mistake. I take full responsibility for any remaining mistakes. If you, the reader, find a mistake, please contact me at Research Gate. I assure you, I am not the arrogant type who gets defensive when someone points out a mistake in my work – it happens far too often for such an emotional response – I just say "thank you" and fix it.

Finally, I want to thank Warren Farrell for alerting me to the boy crisis. I already had anecdotal evidence of this, but it was his book that informed me that boys throughout America are in disarray. I resolved to write a mathematics textbook that boys would enjoy and become enthusiastic about. No more would geometry be just boring vocabulary tests and endless plugging and chugging of memorized algebra formulas for surface area and volume. No! Forty years ago it was high-school geometry that made me decide to become a mathematician (I had previously aspired to be a military officer) and I knew geometry could inspire modern boys as well.

Geometry–Do

Victor Aguilar
Author of *Axiomatic Theory of Economics*

This textbook is at the high-school level, but it is written without the condescending tone typical of American high-school textbooks. It is divided into two volumes, *Geometry without Multiplication: White through Red Belt* and *Geometry with Multiplication: Blue and Black Belt*. Our purpose is to instruct students in logic; thus, their arguments will be sound.

I rank high-school geometry students analogous to the way *Tang–Soo–Do* students are ranked. With these ranks and by addressing the student as Grasshopper, like in the old TV series *Kung Fu* (now on *Nickelodeon*), I hope to motivate students to strive for each successive colored belt. "A soldier will fight long and hard for a bit of colored ribbon," quoth Napoleon Bonaparte.

White Belt	Foundations	Red Belt	Famous Theorems
Yellow Belt	Congruence	Blue Belt	Quadrature Theory
Orange Belt	Parallelograms	Cho–Dan	Harmonic Division
Green Belt	Triangle Construction	Yi–Dan	Circle Inversion

The white-belt chapter teaches the geometry needed by all construction workers. Students learn how to make a foundation square with no auxiliary lines outside it. This is because it may be in a hole if it is for a basement, or it may be surrounded by trees or cliffs if a plot of land was cleared and graded for a house being built in a forest or cut into a hillside. Also, students learn how to construct strong and inexpensive wooden gantries and gates wide enough for farm equipment to go through. Bridges for both pedestrians and vehicles are described, and the basics of fortresses are explained as a lead-in for the machine gun emplacement lessons to come. The yellow-belt chapter provides a theoretical foundation for the later chapters and concludes with geometry needed by architects for designing custom-made mansions, churches, museums, *etc.* The orange-belt chapter teaches geometry needed by asphalt men and the theory needed by architects. The green-belt chapter teaches geometry needed by sea captains and military officers. Many triangle construction problems are solved. There is detailed instruction on navigating a ship with a sextant and on setting ambushes with heavy machine guns. Students are trained for the *International Mathematical Olympiad* (IMO). Red belt presents theorems difficult enough that they went unsolved for decades and are now named after famous mathematicians. We consider the work of Miquel, Wallace, Torricelli, Napoleon, Fagnano, Euler, *et. al.* Students get serious about the IMO, though they cannot expect to win it until they become black belts.

Volume Two begins with the blue-belt chapter, which teaches the quadrature theory needed by surveyors to calculate the area of irregularly shaped farm fields. Multiplication is presented midway through, and the power of a point is defined. With this theorem, the Cramer-Castillon Problem is solved, which my friend Milan Zlatanović found a much better solution to than Castillon did in 1776. Castillon had a lengthy and difficult solution that cites Menelaus' Theorem, which is quite advanced, while Zlatanović has a short and elegant solution. For the benefit of *Volume One* readers, I put his proof in the red-belt appendix, *A Look Ahead: Blue Belt!* After blue belt comes *Cho–Dan* (1st degree black belt), which is harmonic division. *Yi–Dan* (2nd degree black belt) is circle inversion and *Sam–Dan* (3rd degree black belt) is projective geometry. We conclude with an exam of problems taken from past *International Mathematical Olympiad* competitions.

Coxeter (1961) wrote, "For the last thirty or forty years, Americans have somehow lost interest in geometry. The present book constitutes an attempt to revitalize this sadly neglected subject. The four parts correspond roughly to the four years of college work." But, if it was college work, then why did he call it *Introduction to Geometry?* To revitalize geometry, he should have replaced Hall and Stevens' *School Geometry*, which had been in use in Canadian high schools since 1918. My revitalization effort is to bring high-school students up to the point where they can read Coxeter, Altshiller-Court or Johnson when they get to college. A secondary objective is to oppose economists who hate deductive logic. The editor[1] of the *Real-World Economics Review* writes:

> *It is a completely mistaken idea that scientific theory is based on deductions from a series of postulates – that is the description of the methodology of mathematics… There is no science which uses axioms and logical deductions to derive scientific theory.*
>
> *Mathematics is not a "science" since it is not based in any direct way on observational evidence. Unlike scientific laws, mathematical laws are not affirmed observational evidence. Recognition of the possibility that there are bodies of knowledge which are not science would lead to greater tolerance and pluralism which is currently desperately needed.*

Tolerance? No. Such outrageous talk by the *World Economics Association* has brought war and discord to our once quiet study of triangles. The editor of the *Real-World Economics Review* is an influential man, and he wields this terrifying power to no other end than to ban all mention of deductive logic. I was not the one who brought war to the study of geometry but, by teaching it as a martial art, I intend to win that war. Logic alone stands in the path of the WEA!

[1] www3.unifr.ch/econophysics/?q=content/deification-science-its-disastrous-consequences

Preface

In 2000, the *Post-Autistic Economics Network* was founded; it denounced this author and every other mathematician involved in economics as "autistic." In May 2011, they changed their name to the *World Economics Association*. The blacklisting of mathematicians had been going on for some time; Gerard Debreu renounced General Equilibrium in 1974 and blubbered an apology in 1983 at his Nobel awards ceremony, described here[2] as "Debreu's axiomatic method" as though Debreu invented it and that he and it rise and fall together. But Alan Kirman describes a "palace revolution" in 1974 as mathematicians piled on and Debreu backpedaled. He quotes Debreu's 1983 Nobel introduction, "Gerard Debreu symbolizes the use of a new mathematical apparatus," but fails to point out how absurd this must have sounded to mathematicians who had ejected the man ten years earlier for bringing shame to a very old mathematical apparatus.

In 1974, the axiomatic method fell with such suddenness that oxygen masks deployed from the office ceilings of geometry teachers worldwide. In the year or two that it takes to replace high-school textbooks, geometry was no longer taught as an axiomatic science. In 1987, when the U.S. stock market fell, economists – without any evidence – railed against deductive logic as the cause of economic collapse, a refrain that we would hear again in 2008; this[3] is typical rhetoric.

Today, 2016, I hope to bring geometry back to its axiomatic roots, as first employed by Euclid 2300 years past, while carefully distancing myself from that hated man, Gerard Debreu.

Postulate and axiom are synonymous, but I use the former term to refer to those stated by Euclid and the latter term to those added later regarding abstract algebra. Some geometers use these terms to distinguish Euclid's controversial parallel postulate from his others, but I feel that, if we are going to teach Euclidean geometry, then we will refer to them all the same way. The work of Lobachevski, Bolyai and Riemann can exist peacefully alongside our work; consistency is not applicability. Also, I use the term axiom to refer to the foundations of other sciences such as Newton's three axioms of motion, Euler's three axioms of ballistics, Richardson's seven axioms of meteorology, Einstein's two axioms of relativity, or my own three axioms of economics.

But whether called postulates or axioms, the important point is that my theory is deduced from Euclid's five postulates plus one more of my own, the axioms describing equivalence relations, total orderings, additive groups, *and from nothing else!* I do not present – without proof – theorems that can only be proven with calculus in a vain attempt to convince students that I had

[2] Gerard Debreu; legacy: www.researchgate.net/publication/291333961_Gerard_Debreu_Ghostly_Whipping_Boy

[3] The *Real-World Economics Review* editor: rwer.wordpress.com/2014/12/22/the-failure-of-economics-is-due-to-the-use-of-axiomatic-method/ He is criticizing economists from before 1974, when the WEA came to power.

proven them from the axioms of geometry. I do not casually assume the field axioms for real numbers, nor do I employ the method of superposition. Robin Hartshorne (1997, p. 2) writes, "The method of superposition used [by Euclid] in the proof of [Book I, Proposition 4, SAS Congruence], which allows one to move the triangle ABC so that it lies on top of the triangle DEF, cannot be justified from the axioms." I fully agree! I do not employ transformations, which are computer-generated demonstrations of the method of superposition. And I do not use the Similarity/Dilation Axiom; I *prove* the triangle similarity theorem.

Common Core geometry textbooks state the triangle similarity theorem as an axiom, called either the similarity axiom or the dilation axiom, and then state without proof the AA, SAS and SSS similarity theorems. Triangle congruence theorems are then just special cases of the similarity/dilation axiom with the scale (dilation factor) being the multiplicative identity. The mid-segment theorem is a special case of the similarity/dilation axiom with the scale (dilation factor) being half. Along the way, the transversal theorems are also stated without proof.

That is the fastest path through geometry *ever!* But what did the students learn beyond memorizing formulas? They certainly did not learn anything about deductive logic. From their point of view, all these statements are just factoids to be memorized. Learning about proofs is a charade. The fact that their teacher and the author of their textbook have put themselves above the need to prove their statements does not go unnoticed. Demanding that students "prove" their statements on exams by citing these unproven factoids smacks of demagoguery.

The biggest difference between this textbook and *Common Core* textbooks is that I do not put myself above the need to prove what I say and that I carry out these proofs with a small number of clearly stated assumptions. I do not silently assume the field axioms for real numbers; when the time comes (*Volume Two*), multiplication will be formally introduced. Nor do I make grand assumptions like the similarity/dilation axiom from whence every theorem is a special case.

I start with Euclid's five postulates, plus one more of my own; also, I cite the well-known axioms describing equivalence relations, total orderings, and additive groups. I go until I have proven everything that I know how to. Then I stop. That is all; nothing more, nothing less.

Dedication

For his decision to *not* use the lame textbook that he was given, I dedicate this to my old-school math teacher, Mr. Duane Scholl of Kremmling, Colorado. Also, I dedicate this to the authors of *Glencoe Geometry*. If it were not for their insistence that I add lengths to angles, I would not have walked out on a prestigious substitute teacher job and decided to write my own textbook.

Note to People Who Hate Math

As a practical matter, this is everybody in America who is not an engineer. This is an observation that many people have made, but the explanation invariably gets cause and effect backwards. It is generally assumed that engineers enjoyed their high-school math classes and thus decided to major in engineering so they could get even more of the subject. This is a self-serving explanation used by teachers trying to take credit for the occasional student who becomes an engineer. When engineers were in high school, they hated their math classes as much as the next guy. They majored in engineering because of the potential money and/or because their father was not going to pay for college if they majored in some lame-brain course like sociology. The reason they say "yes" when you ask them ten years later if they like math is because they are thinking back to their college math classes, not their high-school math classes, which they still hate.

Why does everybody hate their high-school math classes? Because the textbooks are bloated and condescending. Module One of Houghton-Mifflin-Harcourt's 1250-page *Geometry* is 60 pages and teaches – *Drum roll, please!* – segment addition, finding the midpoint of a segment by folding the paper over, angle addition and angle bisection. The authors (these books are always written by a committee) are making two big assumptions about their readers, both of which are wrong and, indeed, mutually exclusive. Their first assumption is that students read about four years below their grade level. This results in a condescending tone that insults everybody across the board; the strong students, the mediocre students, even the weak students are offended. The second assumption is that students will read 1,250 pages in a single school year. This would not be true even if they read at grade level. Nobody – strong, mediocre, or weak – has ever read 1250 pages on any topic; certainly not in a single school year when they are taking four other classes that have equally bloated textbooks. Old-time textbooks were not bloated like this.[4]

When I entered high school (1979), my math teacher simply announced that the textbook was bloated and condescending, and we would be learning solely from his lectures. The parents complained – Do parents do anything *but* complain? – and asked, if the textbook was so bad, why didn't he buy a different one? "Because they're all like that," he explained. I am a mathematician today *only* because my old high-school teacher refused to teach from the textbook. Had he been like most teachers and used the textbook, I would now hate math just as much as everybody else does. When I got to college, my professor observed that, though I was his top student, I never read the book; he asked why not. "Textbooks are bloated and condescending." "In college, they are not," he said, and he was right. Thus, I now write of high-school geometry in the college style.

[4] Wolf and Phelps' *Practical Shop Mathematics* has an 80-page chapter with 50 theorems that exceeds *Common Core* by about forty theorems. They write, "[it is] of much greater value to the high school student who is not going to college than is the usual geometry course consisting of about 150 theorems" ([1935] 1958, p. v). Here are over 300.

Note to Students

I received 72 credits in the math department as an undergraduate, which was every class that the university offered. But even after all those classes, I can say without exaggeration that high-school geometry was the most enjoyable math class I ever took. Actually, inspiring is probably a better word, because it was geometry that made me decide to become a mathematician.

Why did I find geometry so inspiring? On reflection, I believe that this is because I am basically an inventor at heart. Indeed, I have invented a number of things since graduating, including an economic theory, a cryptosystem, a system to play casino blackjack and a weapon to kill SAM crews on a skyscraper without exposing aircraft to missile fire and without risk to civilians in the streets below. Other math classes taught me a lot of formulas and equations, but only geometry gave me the sense of inventing a self-contained science from the ground up, which is really what it is like – I can tell you from experience – to invent some new weapon or device.

Sadly, since becoming a grown up, no conversation I have had with a high-school student or recent graduate has indicated that they felt this sense of invention when studying geometry. Without exception, they felt that geometry was a big waste of time, a class stuck incongruously between Algebra I and Algebra II that – in actual practice – amounted to nothing more than a review of Algebra I, a subject that they had already demonstrated their mastery of by acing. It is my hope that this textbook will help kids discover the sense of invention that so thrilled me!

A word to the wise, Grasshopper: In the military they say, “train like you fight.” The website that accompanies white and yellow belt has illustrations but, after that, you must draw your own. And you are advised to draw your own in white and yellow belt before consulting the website. Students fail because they get too relaxed when studying. Their only movement is their eyeballs traversing back and forth. They say, “I understand” after reading each proof, but their mind is wandering. Then they fail the exam and say, “I’m bad at exams.” No, they are bad at studying.

You would not last long in the ring if your only experience with boxing was sitting on your butt watching *Monday Night Fights*. And, when I write a geometry problem on the board, set a blank sheet of paper on your desk and say, “Solve it!” it had better not be the first time you have used your compass and straight edge. You would not go into combat with a rifle you have never fired at the range, would you? These are not going to be those pansy exams where they solve the entire problem but leave one theorem citation blank and ask you to fill it in. We just state the problem and hand you a blank sheet of paper. It is up to you to write something intelligent on it.

Scared? Then take *Common Core* geometry; all they ask of you is to memorize formulas. *Loser!*

Using Geometry to Address the Boy Crisis in Education

In America, we have what is known as the boy gap or the boy crisis; this means that boys are falling behind girls in school, especially in reading, but also in their traditional stronghold of math. Besides schools not teaching martial skills, this failure is because schools have been feminized:

1. **Lack of movement.** Boys are more antsy than girls and need to move around, but nobody ever goes outside to perform a geometry construction with stakes and spools of string.

2. **Boring review drills.** Boys enjoy the challenge of high-stakes exams and will work hard to prepare. But, for those facets that they are confident in, they feel that their homework is just needless drudgery; thus, they fall behind in classes where homework is graded.

3. **No edifices to construct.** Boys like to build something that they can hold up for their parents and their peers to admire, while girls are fine plugging away at endless homework assignments. Building an A-frame out of boards and wire rope is a good physical project for white belts. Also, geometry itself is an edifice at an abstract level because it is built on a foundation of postulates; the theorems then pile up like the floors in a skyscraper.

Zero tolerance of violence is going too far; high school is not fight club, but it is not a convent either. Learning martial skills will not turn the boys into psychopathic killers. In *Geometry–Do*, machine gun emplacement is green belt; until then, it is like a carrot in front of a donkey to keep the boys going even in the face of difficulty. Insert a day when you begin your lecture only to be interrupted by the principal running in and shouting that the Russians have invaded, and your class is needed to set an ambush for them. Shove everybody into a school bus and race them to a point below an elevated freeway. Tell them that comes a convoy of trucks and BTR-80 armored personnel carriers (BTR is the Russian equivalent of APC) and that their 14.5 mm autocannons can only be depressed 4°. With the 2° slope of the freeway – to allow water to drain off it – their guns can be depressed 6°. Find an ambush site as far from the freeway as possible to avoid RPG-7 fire from their dismounted soldiers, but close enough to be below cannon fire. The answer is $x = \frac{h}{\tan 6°}$. Just measure the height of the overpass with a laser rangefinder, h, and then use it to lay off the distance to your ambush site, x. Being underneath cannon fire, they should not put all their sandbags in front, but should form a circle to protect from grenades.

Tangent is trigonometry, so this problem is review of the students' 7th grade mathematics when they learned the definitions of the trig functions. But a little review never hurts! In martial societies, every 12-year-old knows by heart the angle of depression of the gun on his enemy's armored vehicles; he has been playing at setting ambushes like this since he was in kindergarten.

Laser rangefinders cost less than \$100 and they are great for getting antsy boys up and moving. Laying off 13.7 centimeters on paper with a ruler and laying off 137 meters in a vacant lot with a laser and marking the endpoints by pounding in sharpened rebar are, conceptually, the same activity. But the latter is fun while the former makes the boys squirm and get accused of ADHD.

The Pisa Tree Problem, among the white-belt practice problems, can be done without the tree. Pretend that you have taken the given measurements, cite SSS to use your laser to construct a congruent $17 : 22 : 32$ meter triangle in the parking lot, and then measure its altitude with your laser. Note that the sun is nowhere cited; this is *not* the classic problem of measuring shadows first performed by Thales to measure the height of the Great Pyramid. Thales' method cites triangle similarity, which is beyond white-belt *Geometry–Do*. Also, constructing a $17 : 22 : 32$ centimeter triangle on paper is not correct; scaling it up to meters requires triangle similarity.

Students should understand that real-life problems are often bigger than a $25'$ tape can measure. But a $100'$ string of the type that carpenters put chalk on can be rotated around a point like a compass, pinched off and carried to another place to transfer lengths, and stretched from one point past another point to extend a segment. You can accomplish a lot with a couple spools of string! Problem 1.26 teaches the geometer's way to square a house foundation. Do it outdoors!

An A-frame is the physical embodiment of most of what is taught to white belts. Build one! It only requires \$20 of lumber, wire rope and screws. Yellow belt ends with a discussion of Tudor arches. Building one is a bit much for a geometry class, but it can be done if you join forces with the shop class, and especially if a boy's father is a carpenter and will help and provide materials. The old man might learn something! Not every professional carpenter knows how to do this. A stone bridge can be modeled by cutting a $4" \times 6"$ board into isosceles triangle frustums and gluing them together to simulate stone construction, like the old-time Tudor bridge builders. A level on a tripod can be used to solve problem 2.4, measuring the distance across a river. Do it at a real river! A surveyor's transit is more expensive, but it can be used to solve problem 2.5.

Get the boys moving, avoid boring review drills, and construct edifices, either real ones made of brick and mortar, or conceptual ones made of postulates and theorems that cite only previously proven theorems. There is no boy crisis in *Geometry–Do*, at least not if it is taught right.[5] Danica McKellar wrote *Girls Get Curves* and boasts of helping "children" succeed in geometry, though she obviously means just the girls. And behind all that boastful talk is a book that teaches no more geometry than what a girl could write on the palm of her hand.[6] I am here to do the same for the boys, but with a *lot* more geometry content. Plus, machine guns! What's not to love?

[5] https://boycrisis.org/ "Boys are 50 percent less likely than girls to meet basic proficiency in reading, math, and science."
[6] Nobody ever accused Danica of leading students on a death march; she's too busy busting glass ceilings to teach actual math.

A Single Page of Formulas Is All It Takes to Pass *Common Core* Geometry

Common Core "geometry" has nothing to do with geometry and everything to do with memorizing obscure algebra formulas that you will never use.[7] The *Varsity Tutors* Advanced Geometry Exam[8] also has tetrahedrons.[9] *Memorization is stupid and boring!* Tell them I said so.

r is the radius of a sphere, a cylinder or a cone. x is a triangle side or a tetrahedron edge.

	Eq. Triangle	Sphere	Cylinder	Cone	Tetrahedron
height	$h = \frac{\sqrt{3}}{2}x$		h	h	$h = \frac{\sqrt{6}}{3}x$
slant length				$l = \sqrt{r^2 + h^2}$	
base area			$B = \pi r^2$	$B = \pi r^2$	$B = \frac{\sqrt{3}}{4}x^2$
lateral area			$L = 2\pi rh$	$L = \pi rl$	
total area	$A = \frac{\sqrt{3}}{4}x^2$	$A = 4\pi r^2$	$A = 2B + L$	$A = B + L$	$A = \sqrt{3}x^2$
volume		$V = \frac{4}{3}\pi r^3$	$V = Bh$	$V = \frac{Bh}{3}$	$V = \frac{\sqrt{2}}{12}x^3$

For kites and rhombi (a rhombus is a kite with equal sides), $A = \frac{pq}{2}$ with p and q the diagonals. The sides of a rhombus are $x = \frac{\sqrt{p^2+q^2}}{2}$. You must know that a trapezoid's area is $A = hw$ with w the semisum of the base and the top (parallel sides). The rest of the exam is basic algebra: To reflect a polynomial over $x = 0$, negate the odd terms; to reflect it over $y = 0$, negate all the terms. Know how to complete the square to get $(x - x_0)^2 + (y - y_0)^2 = r^2$ and the meaning of x_0, y_0, r. Given two points, be able to find the line through them, the distance between them, and their midpoint. Also, find a line through a point given its slope or given a perpendicular line. Know the definitions of sine, cosine, and tangent, but you will not need trigonometric identities.

Isn't that amazing? Without knowing a single geometry theorem, you can now call yourself a *Varsity Tutor* master of "advanced geometry." Thus, if you are a loser who wants only to learn the minimum needed to graduate from an American public high school, then just photocopy this page and return *Geometry–Do* to the textbook store so somebody who is not a loser can buy it.

[7] The reason why *Common Core* is focused on memorizing obscure algebra formulas is obvious: Bill Gates is bribing his way into a monopoly on educational software. Computers are good at plugging numbers into memorized equations; humans not so much. Gates can sell software whose kernel is three equations and 99% user interface.

[8] Try it with just this page! www.varsitytutors.com/advanced_geometry_diagnostic_2-problem-89385

[9] Want to stump a *Varsity* tutor? Ask him for the inradius and circumradius! They are $r = \frac{\sqrt{6}}{12}x$ and $R = \frac{\sqrt{6}}{4}x$.

Note to Teachers

This textbook is intended for both general and honor students in both America and India. It is in American English, but without the condescending tone prevalent in American textbooks. **Military schools** must require a two-year program to have time for machine gun emplacement.

White and yellow belt serve three purposes: It provides a theoretical foundation for the later chapters; it prepares students for a class in non-Euclidean geometry; and it teaches construction workers what they need to know to excel in carpentry, masonry, concrete, and asphalt work. The first two are achieved by teaching only absolute geometry; that is, the theorems common to both Euclidean and non-Euclidean geometry. Orange belt introduces the parallel postulate. This may seem a bit abstract for construction workers, but the difference between intrinsic and metric geometry is illustrated by solving the same problem with two spools of string or with a tape measure, so it is often the construction workers who are first to accept intrinsic geometry.

Everybody: Take at least a full semester going through white and yellow belt, proving every theorem, and solving every problem. For the Americans, this is 11$^{\text{th}}$ grade, and for the Indians, Level IX; the stronger students will have seen many, but not all, of these theorems before, though probably not treated so formally, and the weaker students will appreciate that the accompanying website gives step-by-step explanations. Award an orange belt on completion.

High-School Shop Class: Everything you need to know is white and yellow belt. Stretch it out to a full year by studying geometry three days a week and memorizing the building codes or doing field work at a construction site two days a week. Award an orange belt on completion.

Catholic Students: Historically, the building of cathedrals and abbeys was a principal motivator for the study of geometry. *Orange Belt Geometry for Construction Workers* at the end of yellow belt discusses such architecture. Thus, follow the same program as high-school shop students but with less theory and more history, supplemented by photos or – if you can afford it – a field trip to Europe. Advanced yellow belt is mostly the work of Saccheri, who was a Jesuit priest, as was Viviani, taught in early orange belt. Students learn what it was like to be a Catholic *c.* 1700. Torricelli and Ceva were also Jesuits, though their work waits for red and blue belt, respectively.

Home School Students: Religious children kept home because their parents fear immoral public schools should follow the Catholic program. Children of engineers and scientists kept home because their parents are contemptuous of public-school teachers should follow the same program as Indian students attending a school where English is the language of instruction. Expelled students are probably going to wind up in the army; I can help them make it their career. Follow the military academy program, learn machine gun emplacement, and try to be a sergeant.

***Common Core* Students**: Spend 2nd semester memorizing all those theorems labeled **CC** (*Common Core*) in the index. Finish with *Single Page of Formulas* and *Squares and Rectangles and Rhombi! Oh My!* and *How to Take Standardized Exams that Define Geometry in Terms of Motion*. Award a green stripe on their orange belts since they did not become real green belts.

American Honor Students and All Indians: Spend 2nd semester on orange belt; award a green belt on passing the green-belt entrance exam. Yellow and orange are shades of white, but green is intermediate! If students pass the class, but not the exam, a green stripe on their orange belts.

Second–Year Geometry: Those with green stripes on their orange belts retake the green-belt entrance exam. If they fail a second time, then goodbye. Green belt requires either going to sea and plotting a course with a sextant or laying some machine guns to defend one's town. Award a red belt to those who complete green belt and a blue belt to those who complete red belt. Red belt is needed for black belt, but not for blue belt. If you have no plans for a third year, you may consider skipping the red-belt chapter and studying blue-belt quadrature in the fourth semester.

Third–Year Geometry: Blue belt (quadrature) and *Cho–Dan* (harmonic division) are completed, and some of *Yi–Dan* (circle inversion). For the Indians, this is in high school, Level XI and XII. For the Americans, it is in college. Until *Volume Two* is written, I recommend Altshiller-Court (2007) or Johnson (2007). A fourth year completes *Yi–Dan* and teaches *Sam–Dan* (projective geometry).

Geometry–Do advances a lot faster than is typical of American textbooks. In sixty pages we have proven over sixty named theorems. Module One of Haughton-Mifflin-Harcourt's 1250-page *Geometry* is sixty pages and teaches – *Drum roll, please!* – segment addition, finding the midpoint of a segment by folding the paper over, angle addition and angle bisection. *Woo hoo!* But *Geometry–Do* students do sometimes fall behind. A principal cause is that they did not learn vocabulary words *before* each lecture. The terms in the glossary are color coded to the chapters where they are introduced. New terms are in **boldface**, but I do not pause to define them. It is your job to tell students at the end of each day which terms to look up in the glossary for the next lecture. But do not demand that students learn vocabulary for chapters beyond them. A principal reason why so many American students hate geometry is because it has been taught for the last half century as a vocabulary test. Words like orthocenter that were employed in the days of Wentworth lost all use and became just vocabulary words – ghosts of textbooks past. Thus, the single best piece of advice that I can give teachers and home-school parents to keep their students moving forward at an acceptable pace is to demand that the students learn the necessary vocabulary words – and none of the unnecessary words – on their own the night before each lecture, so the teacher need not spend lecture time defining terms. Homework should always be *preparation* for a lecture, never just a review of what was taught in past lectures.

Why Do Teachers Drop Out of the Profession?

It takes five years to get a teaching certificate, yet many – maybe most – teachers do not last five years on the job before they quit their chosen profession. This is why schools are scouring America for subs, whom they now call paraprofessionals, apparently in an attempt to entice people who do not know what the prefix *para-* means. They will hire *anyone* to teach math!

> *At the grocery store, at Target, at Starbucks, anywhere I go, if I meet someone who seems smart and engaging, I give them my card and say, "Be a teacher! It doesn't matter what your circumstances are."* – Traci Taylor[10]

To hear the teacher's union tell it, money is what is needed. If taxpayers will just approve another school bond, everything will be fine. This is not true. Teachers make the same as tradesmen. Both have the potential to make $60K a year at the time of retirement. The carpenter who starts at $25K a year does so; the certified teacher who starts at $35K a year quits to become a carpenter; the sub who starts at $15K a year quits to become… well, any job is better than that.

Surveillance cameras emasculate the teacher. Every decision that a teacher makes is subject to a supervisor pulling the video and then casually reversing the teacher's decision, and doing so right in front of all the students to make sure they know who is in charge; it is not the teacher. "Empowered" is just a buzzword. When all the teachers, both male and female, are holding a meeting in the men's toilet to dodge the surveillance cameras, nobody is feeling too empowered.

Sports stars can demand passing grades. If a football player flunks a geometry test a week before the big game, there is a *lot* of pressure on the principal to fix this, and – as any plumber knows – feces flows downhill. The teacher, being the low man on the totem pole, just changes the grade.

Content experts are not respected. I majored in math and yet I never met an aspiring high-school teacher in any Math Department class. Their five-year program was all Education Department classes. They did not study any mathematics – just endless, mind-numbingly boring **edubabble**. Many certified teachers drop out from embarrassment – they cannot face 16-year-old boys who knows more math than they do – while the survivors just learn to hate and fear mathematicians.

Drug use is a survival mechanism. In high school, aspiring teachers were filled with idealism; they sang the XCX song backwards, "I don't wanna break the rules, I just wanna go to school." But, in college, the mind-numbing boredom of edubabble lectures changed them. They could see a door opening and they knew what was behind it – insanity. Weed helped. As teachers, they graduated to pills when the boredom of monitoring standardized tests broke them. Then they got busted.

[10] www.edsource.org/2017/fresno-tackles-its-shortage-of-math-and-science-teachers/581342

What Should Replace Common Core?

Common Core proponents sneer at their opponents[11] for having no goal or some goof-ball goal.

1. A few *Common Core* opponents are religious zealots that are *still* trying to deny evolution.

2. Most of them are housewives who think they know it all when it comes to elementary education, but they become mysteriously silent when high-school education is discussed.

In sharp contrast, the proponents of *Common Core* know exactly what "defeat" would mean: a new paymaster – some billionaire other than Bill Gates trying to bribe his way into a monopoly on educational software. But nobody is as rich as Bill Gates, so they know this is not happening.

One cannot help but notice that the group whose opinions are never sought are the university professors that will be receiving these so-called "college ready" students. Indeed, content experts, as they are called, have about as much say regarding the curriculum as the high-school janitor does. This point is often concealed by *Common Core* proponents' boastful talk of their PhDs. But their PhDs are not in the subjects that they are teaching; they are PhDs in education.[12]

Who controlled the curriculum in the 19th century? Wentworth ([1868] 1899, p. 180) writes:

> ***Proposition XXIII*** *The square on the bisector of an angle of a triangle is equal to the product of the sides of this angle diminished by the product of the segments made by the bisector upon the third side of the triangle.*
>
> ***Proposition XXIV*** *In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle by the altitude upon the third side.*
>
> *This theorem may be omitted without destroying the sequence. Props. XXIII and XXIV are occasionally demanded in college entrance examinations, but they are not necessary for proving subsequent propositions or for any of the exercises. Teachers may therefore use their judgement as to including them.*

Geometry–Do proves both theorems, but the point is that 19th century professors controlled the high-school curriculum by writing entrance exams. Sadly, such entrance exams no longer exist.

[11] Missouri budgeted billions of dollars for the implementation of *Common Core*, mostly going towards the purchase of software being sold by you-know-who, and $8 for tin foil hats to give to the opponents of *Common Core*.

[12] David Conley has a BA in social sciences, an MA in multiculturalism and a PhD in school administration. He has never taken a college-level mathematics class in his life. The only actual teaching experience of this pasty-faced multiculturalist is an Ethnic Heritage Program in Jefferson County Public Schools from 1978 to 1982.

Defeat of *Common Core* should be easy since almost everyone hates it, the only exceptions being a few paid shills of software moguls and a few simpletons who are dazzled by buzzwords like "high achieving" or "college ready." Admittedly, David Conley is one of the slickest propagandists around; second only to the Trolls from Olgino, nobody can spit out buzzwords like Conley!

The problem is that *Common Core* opponents do not know what they want. Here are some goals:

1. Eliminate the SAT and any other nationwide high-school test. The SAT is the root cause of corruption in education; it attracts software moguls like dead cows attract flies. Also, it is impossible for a single test to judge those who plan to attend a trade school, a divinity school, a state university, or a prestigious private college. Every institution of higher learning should have its own entrance exam written by its own professors. That is how it was in Wentworth's day, and America was on top of the world back then. Now we import all our techies from India while American boys vacuum the floors in their offices.

2. Eliminate Capital-E Educators and revoke every education degree that has ever been granted. If a college student wants to teach in high school, he *must* major in the subject that he intends to teach. To teach high-school mathematics, one must have a B.S. in mathematics; to teach high-school chemistry, one must have a B.S. in – *You guessed it!* – chemistry. No education majors! It is a vacuous degree given to brainless people. We will never fix education in America if the word "qualified" refers to Capital-E Educators, as it does now, but *not* to people who majored in the subject that they intend to teach.

"Almost one third of all high school math teachers have neither a major nor a minor in math or a related field," notes Heather Voke[13], "One-fourth of all beginning teachers leave the classroom within the first four years… Even more alarming than the turnover rates themselves are data suggesting that the most intelligent and effective teachers leave the profession at the highest rates… new teachers who scored in the top quartile on their college entrance exams are nearly twice as likely to leave teaching than those with lower scores."

Why are they leaving? Voke cites Joel Spring, "In recent years the satisfaction that teachers have gained from autonomous decision making and creativity has been threatened by expanding bureaucratic structures and attempts to control teacher behavior in the classroom."

Why did I leave teaching? Rather than teach page 256 of *Glencoe Geometry* that is adding lengths to angles, I walked out. I have too much respect for mathematics to subvert it for a job.[14]

[13] www.ascd.org/publications/books/104138/chapters/Responding-to-the-Teacher-Shortage.aspx
[14] www.researchgate.net/publication/320696944_How_Algebra_But_Not_Geometry_Is_Based_on_Happy_Coincidences

The NES formula sheet[15] might explain how Traci Taylor can recruit "math teachers" at Starbucks! (She means *substitute* teachers, but she talks fast so recruits do not notice "substitute" missing.)

Ware the NES document!!! It is here to be criticized! Do not *use it for reference later in the book!*

NES Profile: Mathematics (304)

SECONDARY MATHEMATICS FORMULAS

Formula	Description
$V = \frac{1}{3}Bh$	Volume of a right cone and a pyramid
$V = Bh$	Volume of a cylinder and prism
$V = \frac{4}{3}\pi r^3$	Volume of a sphere
$A = 2\pi rh + 2\pi r^2$	Surface area of a cylinder
$A = 4\pi r^2$	Surface area of a sphere
$A = \pi r\sqrt{r^2 + h^2} = \pi r\ell$	Lateral surface area of a right cone
$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + a_n)$	Sum of an arithmetic series
$S_n = \frac{a(1-r^n)}{1-r}$ Wrong!!!	Sum of a finite geometric series This needs the condition $r \neq 1$
$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \lvert r \rvert < 1$	Sum of an infinite geometric series
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	Law of sines Is c the side of the triangle or $\frac{1}{4}$ of the latus rectum as shown below?
$c^2 = a^2 + b^2 - 2ab\cos C$	Law of cosines
$(x-h)^2 + (y-k)^2 = r^2$	Equation of a circle
$(y-k) = 4c(x-h)^2$ Wrong!!!	Equation of a parabola $(y-k) = \frac{1}{4c}(x-h)^2$
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	Equation of an ellipse
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	Equation of a hyperbola

The *National Evaluation Series* (NES) are tests for prospective teachers in each high-school subject; 304 is mathematics. The fifteen formulas are all that are required to become a high-school math teacher and they do not even have to be memorized; the formula sheet can be taken into the exam. *The parabola equation is wrong!!!* When done right, c is the distance from the vertex to either the focus or the directrix; $4c$ is the length of the latus rectum. Also, it is stupid to use c (not w) for this *and* for the constant term in $y = ax^2 + bx + c$. It is no surprise to me that new teachers who scored in the top quartile on their collage entrance exams are nearly twice as likely to leave teaching than those with lower scores. They are appalled to discover that *Common Core* is teaching garbage!

[15] This was written by an "educator," not a mathematician: www.nestest.com/Content/Docs/NES_Profile_304.pdf

Preliminary Exam for the Experts

Geometry–Do does not prerequisite any prior knowledge of geometry; indeed, Algebra I is strictly sufficient, though the extended chains of reasoning will be met most successfully by students with a bit more mathematical maturity than that. The following exam is for experts with prior knowledge of geometry to assess where their expertise lies in the *Geometry–Do* hierarchy.

White Belt

You are an army captain in command of five platoons and tasked with defending a 16-klick straight segment of the border. Position four platoons in bases with high watchtowers and the fifth hidden behind them to act as a reserve. Not having a computer with graphic-design software and a color printer in your tent, you must use a compass and straightedge to locate the bases on a paper map. The assumption is that your troops and the enemy troops move at equal speed; the requirement is that they must be met before they get behind your forward bases. Also, the bases have mortars, so you must draw a line demarcating the free-fire zone where your troops need not go to get in front of an enemy incursion. The fifth base has the same assumption and requirement, though it covers the entire 16-klick front in case the enemy gets past your line.

Yellow Belt

Your house is some distance from a long straight highway leading to a town. Construct a curved driveway that is tangent to the highway, of constant turning radius, and on your own property. This will allow you to accelerate on the driveway to safely merge with the highway traffic.

Two towns are on the same side of a straight railroad track and some distance away. Where should a railway station be built to minimize the sum of the roads to the two towns?

There is a roughly circular lake, a straight highway, and an abandoned farm. You have purchased the farm with the idea of turning the farmhouse into a way station for fishermen. Pave a straight road to the lake so the farm is at its exact midpoint.

You have been tasked with constructing a sixteen-meter wide Tudor arch at the front gate leading to a mansion. Write detailed instructions for the carpenter on how to build the concrete forms.

Orange Belt

Two country roads intersect at an arbitrary angle. Pave an arc connecting them and going around the corner of a farmer's field, which is on the angle bisector of the two roads.

From a house in the country, construct a dirt road to a straight paved road, the latter twice as fast as the former, to minimize travel time to a nearby town on the paved road.

A river with parallel banks passes between two towns at an arbitrary angle. Connect the towns with a minimal length road; the bridge must be perpendicular to the river.

Green Belt

A straight fence defines the border between two countries, and, on the enemy side, geographic features define a bottleneck. Given a top traverse of $30°$, the kill chord drawn across the bottleneck defines an arc that your gun must be positioned on. There is a point on the fence that one infantry platoon is tasked with defending to the left of and another platoon to the right of. To avoid the appearance of favoring one platoon over the other, you wish to position your gun so its field of fire covers equal segments of fence to the left and to the right of this point.

The enemy has three antiaircraft guns in an equilateral triangle with a munitions dump at the center. Afraid to attack from the air, you are sneaking up on it with a self-propelled mortar. You are afraid to reveal your position with a laser rangefinder, but you plan to aim over the munitions dump and then walk your shells back until you hear a secondary explosion. How do you aim it?

Red Belt

The enemy has a base with three guns that form an acute triangle; their barracks is inside it. Hit it with three guided bombs to meet these conditions: Every part of the triangle is struck by shrapnel from at least one bomb, and the enemy barracks is struck by shrapnel from every bomb. The bombs are of the same type and thus they have equal-size circles of shrapnel, but to avoid injuring friendly troops poised nearby to overrun the base, the bombs are as small as possible.

A hospital consisting of three big buildings is being built. A diesel generator will provide electricity in the event of a power outage. You have been hired to dig trenches and lay cables to bring power from the generator to each building. Where should the generator be positioned to minimize the total length of cable, and thus also the electricity lost to resistance in the cables?

You are a colonel in command of three army bases at distances of $10, 12$ and 16 klicks from one another. The general wishes to construct a munitions dump inside your triangle and to further defend it with three antiaircraft guns that form an equilateral triangle with the munitions dump at its center and your three bases on each of the three sides of the equilateral triangle. He insists that this equilateral triangle be as large as possible. Locate the guns and the munitions dump.

Table of Contents

The way that you wander, is the way that you chose,
The day that you tarry, is the day that you lose,
Sunshine or thunder, a man will always wonder,
Where the fair wind blows, where the fair wind blows.

If you are coming to this table of contents now for the first time, it may seem as though a black belt in *Geometry–Do* is a mountain too high to climb. But I tell you, the next three years are going to pass anyway, so why not go for the gold? It is an accomplishment you can boast about for the rest of your life. What else can you do as a teenager that you will be proud of as an old man?

A Few Words About Your Kit

The adventure begins! But, before we embark, a few words about your kit. Pentel and Staedtler products are available in office supply stores; Alvin is available only in drafting supply stores.

1. Pentel P205 two-pack 0.5 mm mechanical pencils $\$11^{\underline{79}}$
2. Staedtler universal compass (*not* their student compass) $\$5^{\underline{29}}$
3. Staedtler Mars plastic tubular eraser holder $\$2^{\underline{49}}$
4. *Geometry–Do* center-finding and parallel-finding metric ruler Included

Old-style graphite rods used by draftsmen require constant sharpening and their lines are too faint to be scanned if you are e-mailing homework; I do not recommend them. If you buy a rolling ruler, draw a line, mark zero on it and then roll the ruler back and forth a dozen times to see if it returns to its starting position. Perform this test on the store counter before you accept the ruler.

Rolling rulers can be quite handy and I have one that I use frequently, but it is in inches. The only rolling ruler that is both metric and center-finding is the Alvin 295M, but I tested two of them and they were both defective. So, I invented the *Geometry–Do* ruler, which is metric and center-finding. It does not roll, but it has four perpendicular scales that allow one to find a parallel line through a point by equating the lengths of two perpendiculars. It also allows one to quickly draw an $8 : 10 : 12$ cm triangle, which makes locating the midpoints easy; and there are holes in the ruler to locate this triangle's circumcenter, medial point, orthocenter, incenter and one excenter.

If you get a job in geometry, three decimal digits of accuracy requires drawing larger circles.

Alvin 702V universal compass with detachable beam $\$20^{\underline{70}}$

This compass allows one to clamp a mechanical pencil into it, which makes for darker lines than the old-style graphite rods, and it is large enough to draw 32 cm diameter circles, or 54 cm with the detachable beam. I use the Alvin 702V for whole circles, but I also keep a friction compass handy for the short faint arcs needed to bisect an angle or to construct a segment's mediator.

This is old school. Many geometers use computer software for all their figures and never draw anything with a pencil. But, because of the big money that Bill Gates is throwing around to bribe his way into dictating the geometry curriculum, I do not advocate any computer software to avoid being seen as a shill for some software mogul. If you want to use software, then do so, but beware of getting locked into a situation where geometry ceases to exist the moment that you rise from your school computer. If you cannot bring it into the real world, then what good is it?

Euclid's Postulates Plus One More

Segment	Two points fully define the **segment** between them.
Line	By **extending** it, a segment fully defines a **line**.
Triangle	Three **noncollinear** points fully define a **triangle**.
Circle	The center and the **radius** fully define a **circle**.
Right Angle	All **right angles** are equal; equivalently, all **straight angles** are equal.
Parallel	A line and a point not on it fully define the **parallel** through that point.

Segments are denoted with a bar, $\overline{EF}$; **rays** with an arrow, $\overrightarrow{EF}$, which have endpoint E and are extended on the F side infinitely; lines with a double arrow, $\overleftrightarrow{EF}$, which are extended infinitely both ways; and **angles** as $\angle EFG$ or $\angle F$ if there is only one angle at F. Triangles and quadrilaterals are also denoted with bars, as $\overline{EFG}$ and $\overline{EFGH}$. The **postulates** are in terms of **fully defined**, which means that a **figure** with the given characteristics exists, and it is unique. **Under defined** means figures with the given characteristics are legion. John Playfair stated the parallel postulate as I and David Hilbert do, which is **equivalent** to Euclid's Fifth Postulate (Euclid, 2013, p. 2).

> *If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

While Hilbert and I both found Euclid's postulate to be convoluted and chose Playfair's version, and we both reject real numbers as unsupported by our postulates, we otherwise are different.

Euclid also had five "common notions," which vaguely describe what modern mathematicians call equivalence relations, total orderings, and additive groups.

Equivalence Relations and Total Orderings

A **relation** is an operator, $\mathcal{R}$, that returns either a "true" or a "false" when applied to an ordered pair of elements from a nonempty set. (We only use binary relations, so we can omit "binary.") Relations must be applied to objects from the same set. For instance, $\overline{EF} = \angle G$ is neither true nor false; it is incoherent. There are four ways that relations may be characterized. For one to hold, it must apply to all possible choices x, y, z from the given set, not just some of them.

Reflexive	$x \mathcal{R} x$
Symmetric	$x \mathcal{R} y$ implies $y \mathcal{R} x$
Anti-Symmetric	$x \mathcal{R} y$ and $y \mathcal{R} x$ implies $x = y$
Transitive	$x \mathcal{R} y$ and $y \mathcal{R} z$ implies $x \mathcal{R} z$

A reflexive, symmetric, and transitive relation is called an **equivalence relation**. The principal equivalence relations considered in geometry are **equality**, $=$, which applies to segments, angles, or areas; **congruence**, $\cong$, which applies to triangles; **similarity**, $\sim$, which applies to triangles; and **parallelism**, ∥, which applies to lines. $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ means that $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ do not intersect. There are an infinity of points in the plane; strange and useless results can be made of small finite sets.

Since segments are known only by their **length**, $\overline{EF} = \overline{GH}$ means that $\overline{EF}$ and $\overline{GH}$ are the same length. Since length is the same regardless of direction, it is always true that $\overline{EF} = \overline{FE}$. But triangles are known, not by just one magnitude, but by six. The vertices are ordered to show which ones are equal. $\overline{EFG} \cong \overline{JKL}$ implies $\overline{EF} = \overline{JK}$, $\overline{FG} = \overline{KL}$, $\overline{GE} = \overline{LJ}$, $\angle E = \angle J$, $\angle F = \angle K$ and $\angle G = \angle L$; and these equalities imply congruence. Beware! Writing the vertices of a triangle out of order is one of the most common mistakes made by beginning geometers.

A **quadrilateral** is a union of two triangles adjacent on a side such that it is **convex**; congruence or similarity holds if and only if both pairs of triangles are congruent or similar. If $\overline{EFG} \cong \overline{JKL}$ and $\overline{EHG} \cong \overline{JML}$, then, $\overline{EFGH} \cong \overline{JKLM}$. Analogously, if $\overline{EFG} \sim \overline{JKL}$ and $\overline{EHG} \sim \overline{JML}$, then, $\overline{EFGH} \sim \overline{JKLM}$. Similarity is defined as two triangles with all corresponding angles equal, so $\overline{EFG} \sim \overline{JKL}$ and $\overline{EHG} \sim \overline{JML}$ means that six pairs of corresponding angles are equal. This is more than just saying that the four corresponding **interior** angles of $\overline{EFGH}$ and $\overline{JKLM}$ are equal; thus, it is not true that proving these four angles equal is sufficient to prove $\overline{EFGH} \sim \overline{JKLM}$. A counter-example is a right square and rectangle; they have all right angles, but they are not similar. "Four-sided figure" is a vacuous quadrilateral definition that leads beginners to err by claiming that right squares and rectangles are similar. We make quadrilaterals a logical extension of triangles.

Relations that are anti-symmetric can only be defined if we have already defined equality, because equality is referenced in its definition. (Equality is the only relation that is both symmetric and anti-symmetric.) A relation that is not symmetric but has the other three characteristics is called a **total ordering**. The adjective total is redundant because we said relations must hold for every pair of elements. (Partial orderings, such as subset, exist in other branches of mathematics.) Geometers only use less than or equal to, $\leq$. ($\geq$ could be, though we usually order from small to large; $<$ and $>$ are irreflexive and so are not orderings.) A nonempty set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a **magnitude**. Geometers consider three magnitudes: lengths, angles, and areas.

Note that our definition of magnitude does not imply that real numbers can be associated with lengths, angles, or areas; only that the relations $=$ and $\leq$ exist and have the required properties. (In real-life applications I use integer lengths, denoted by absolute value, *e.g.,* $|\overline{EF}| = 5$ m.) It does imply that magnitudes are unique, which is what the replication **axiom** below is stating.

Equal magnitudes are an equivalence relation and can be reproduced wherever needed; that is, compasses do not collapse when lifted from the paper but are like holding a chain at a length. Compasses that collapse would be like surveyors who can walk a chain around an **arc** but, the moment the center guy moves, their chain turns to smoke. This is a parlor game, not a science!

An **equivalence class** is defined as a subset of all the elements that have an equivalence relation with each other. It can be shown that any two equivalence classes either coincide or are disjoint, hence the collection of equivalence classes form a partition of the set. For example, if the set is all the lines in the plane, it is partitioned by parallelism; each equivalence class is composed of lines stacked on top of each other (parallel) but tilted relative to the lines in the other classes. Equivalence classes can be defined in reference to an existing equivalence class. For instance, if an equivalence class is defined as all the angles equal to a given angle, then all the angles **complementary** to any member of that class are equal to each other; that is, they form their own equivalence class. All the angles **supplementary** to any member of that class are also equal to each other. If an equivalence class is defined as all the lines parallel to a given line, then all the lines **perpendicular** to any member of that class are parallel to each other. All the circles with radii equal to any member of an equivalence class of equal segments are an equivalence class.

Equivalence also refers to statements that can be proven if the other one is assumed, and in either order. For instance, Euclid's fifth postulate and Playfair's postulate are equivalent because, assuming either to be true, it is possible to prove that the other is true. The equivalence of **theorems** can be expressed by separating them with the phrase "if and only if," which can be abbreviated "iff." Proof in the other direction is called the **converse**; that is, if p implies q, then the converse is that q implies p. If p and q are equivalent, then both implications are true.

Proof by contradiction when there is only one alternative that must be proven impossible is called a **dichotomy**. A **trichotomy** (*e.g.* ASA congruence) has three alternatives. A magnitude can either be less than, equal to or greater than another, and only one of these three is desired; thus, by proving the other two to be impossible, we know that it is the one that makes the theorem true.

Additive Groups

We define an additive group as a nonempty set that is closed under an operation that we will denote $+$ and which has these properties for all x, y, z that are members of that set:

Associative property	$(x+y)+z=x+(y+z)$
Commutative property	$x+y=y+x$
Existence and uniqueness of an identity	$x+0=x=0+x$
Existence of unique inverses (identity is its own)	$x+(-x)= \ 0 \ =(-x)+x$

There exist magnitudes that are not additive groups, such as economic value. Given a choice between x or y, it is always possible for a person to choose one. But, because x may substitute for or be a complement to y, they are not independent the way geometric magnitudes are. There are also additive groups that cannot be ordered, such as matrices. Matrices of the same dimension are an additive group, but we cannot say $\boldsymbol{X} \leq \boldsymbol{Y}$ for any two distinct matrices.

On the first day of class I ask the students to look back to a time eight or ten years prior, when they were little kids and knew only how to add and subtract; multiplication and division was still scary for them. I assure them that geometry will be like going back to 1st grade. Sticking segments together end to end or angles together side by side is no more difficult than 1st grade problems about adding chocolates to or subtracting chocolates from a bowl of candies. How easy is that?

Replication Axiom

Given $\overline{EF}$ and $\overrightarrow{JK}$, there exists a unique point L on $\overrightarrow{JK}$ such that $\overline{EF} = \overline{JL}$.

Given $\angle EFG$ and $\overrightarrow{KJ}$, there exist rays $\overrightarrow{KL}$ and $\overrightarrow{KL''}$ such that $\angle EFG = \angle JKL = \angle JKL''$.

The symbol $<$ is defined by the terms "between" and "inside," as stated in the two axioms below. But this symbol can also be applied to magnitudes. $|\overline{EF}| < |\overline{JK}|$ means that the number of units that can be laid off inside $\overline{EF}$ is less than the number that can be laid off inside $\overline{JK}$. The absolute value signs denote these numbers, so we use $+$ when combining them; $|\overline{EF}| + |\overline{JK}|$ is the sum of these lengths. Degrees or radians measure angles and can be added, but that is undefined in this book; it is trigonometry. We measure area though; $|\overline{EFG}| + |\overline{JKL}|$ is their combined area.

Interior Segment Axiom

*If M is **between** E and F, then $\overline{EM} < \overline{EF}$ and $\overline{MF} < \overline{EF}$ and $\overline{EM} \cup \overline{MF} = \overline{EF}$.* (*$\cup$ means union.*)

Interior Angle Axiom

*If P is **inside** $\angle EFG$, then $\angle EFP < \angle EFG$ and $\angle PFG < \angle EFG$ and $\angle EFP \cup \angle PFG = \angle EFG$.*

To be between E and F means to be on the segment they define, $\overline{EF}$, but at neither **endpoint**. To be inside $\angle EFG$ (not straight) means to be between points on $\overrightarrow{FE}$ and on $\overrightarrow{FG}$, with neither point being F. It is instinctive that all humans know what it means for a point to be between two points and – in the case of Pasch's axiom – also what it means for a segment to be continuous; that is, with no gaps where another segment might slip through. Triangles and quadrilaterals are defined to be convex; this means that they are not allowed to be concave or **degenerate**. Interior angles are greater than zero and less than straight (indeed, all angles are because of "between" in the definition), so triangles are never segments, and quadrilaterals are never triangles or darts.

Pasch's Axiom

If a line passes between two vertices of a triangle and does not go through the other vertex, then it passes between it and one of the two vertices.

In *Geometry–Do*, plane, point, shortest path and straight are **undefined terms**. These are concepts that a parent does not have to explain to a child; they are just giving names to what is already in the child's mind. Specifically, a plane is undefined because rigorously defining uncountably infinite, flat, and of exactly two dimensions is beyond the scope of this book. Euclidean **area** is defined as the measure of the size of a triangle or a union of **disjoint** triangles. Like the ancients, we do not have a rigorous definition of limits but just rely on intuition; wheat plants are infinitesimal compared to fields, so weighing the wheat is almost like calculating a limit. Thus, area too is something that small children can understand without explanation. Defining area as the product of a right rectangle's sides waits for *Volume Two: Geometry with Multiplication*. This definition of area is not intuitive to small children, who know nothing of multiplication. For now, just know that area is a magnitude.

Degrees of angle or radians will not be defined in either volume because doing so is trigonometry.

Triangle Inequality Theorem ***(Euclid, Book I, Prop. 20, 22)***

Three lengths can be of triangle sides if and only if the sum of the lengths of any two sides is greater than the length of the third side.

In ancient Greece, Epicurus scoffed at Euclid for proving a theorem that is evident even to an ass (donkey), who knows what the shortest path to a pile of hay is. Some textbooks call it an axiom, and some prove only one direction – they start with the existence of the triangle and prove the inequalities – but that is not the direction needed for SSS, which cites it. Beginners here should just take it as an axiom; also, they should take the continuity theorem (below) as an axiom. Its proof requires the Cantor axiom, which assumes a knowledge of set theory that is not expected of beginning geometers. Experts can find detailed proofs in an appendix at the end of this book.

Continuity Theorem

1. *A line that passes through a point inside a circle intersects the circle exactly twice.*
2. *A circle that passes through points inside and outside a circle intersects it exactly twice.*

The foundations explained above are sufficient through red-belt study. In these early chapters, students will learn to bisect, trisect and quadrisect a segment, and to multiply it by small natural numbers by using repeated addition. No more of these repeated additions are needed than four, for construction of the Egyptian or $3:4:5$ right triangle, except that we mention in passing the

$5 : 12 : 13$ right triangle, which is used by plumbers when installing $22.5°$ elbows. Elementary school teachers are wrong when they define multiplication as repeated addition; this is why so many students are later confounded by real numbers like $\sqrt{2}$ or π. The repeated addition used in $3 : 4 : 5$ right triangles has nothing to do with multiplying lengths as defined in *Volume Two*.

Blue belts will learn of similarity and prove the triangle similarity theorem. They will go beyond bisecting and trisecting segments to constructing segments whose length relative to a given unit is any rational number. Another axiom is needed for this. A nonempty set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a magnitude. But to construct segments whose length relative to a given unit is any rational number, length must also be Archimedean.

Archimedes' Axiom
Given any two segments $\overline{EF} < \overline{GH}$, there exists a natural number, n, such that $n|\overline{EF}| > |\overline{GH}|$.

This may seem trivially true, but Galois (finite) fields are not Archimedean. Every schoolboy is taught that Archimedes claimed that, given a long enough lever and a fulcrum to rest it on, he could move the world. They typically receive no clear answer from their teacher on why it matters, since no such fulcrum exists, and Archimedes seems to ignore that gravity is attractive. The point that Archimedes is making is that, if there were such a fulcrum and much gravity under it, he would need a lever 6×10^{22} longer on his side of the fulcrum to balance his mass against the Earth. If the fulcrum were one meter from Earth, Archimedes would be in the Andromeda galaxy if he stood on the other end of that long lever. 6×10^{22} is a big number, but it does exist.

We said above that undefined terms are concepts that one does not have to explain to a child; the adult is just giving names to concepts that are already in the child's mind. But defining natural numbers as $1, 2, 3, \ldots$ is only intuitive up to as many fingers as the child has. We think 6×10^{22} exists because countably infinite fields are consistent; but so are big Galois fields. This axiom is why it is traditional in America to tell children that every snowflake is unique; it helps them visualize big numbers. (Dinosaurs help them visualize vast gulfs of time.) That Archimedes' axiom is not intuitive to small children is one reason why similarity is delayed until blue belt.

But these are issues of concern to black belts; first, the student must take a short jog through the colored belts, which are concerned with what Mihalescu (2016) refers to as the remarkable elements of triangles and quadrilaterals. By this we initially mean the principal triangle centers. The medians intersect at the medial point, the angle bisectors intersect at the incenter, the altitudes intersect at the orthocenter, and the mediators intersect at the circumcenter. In this introduction the student does not need to know what any of these things are, only that medians – the segment from a vertex to the midpoint of the **opposite** side – and the bisectors of vertex angles are always inside their vertex angle. In $\overline{EFG}$, if the vertex is F, then they are inside $\angle EFG$.

Crossbar Theorem

A ray from a triangle vertex that is inside this angle intersects the opposite side inside of it.

The **infoot** (plural: infeet) is where an angle bisector cuts the opposite side of a triangle; in $\overline{EFG}$ they are denoted E^*, F^*, G^*. Since the angle bisector is inside the angle – indeed, it is exactly halfway inside – the crossbar theorem implies that infeet are always inside the sides of a triangle; they are never at a triangle vertex or on the extension of a side like altitude feet might be. Also, if a point is **visible under an angle**, then the ray from this triangle vertex that passes through this point cuts the opposite side of the triangle inside it, not at its endpoint or on its extension.

Proof of the crossbar theorem is deferred to the appendix, *Foundations of Geometry Revisited*.

The midpoints of segments[16] are denoted by the letter M with a double subscript, which are the endpoints of the segment. Thus, two medians of the triangle $\overline{EFG}$ are $\overline{EM_{FG}}$ and $\overline{FM_{GE}}$. Consider the triangle $\overline{EM_{FG}G}$. The line $\overleftrightarrow{FM_{GE}}$ passes between the vertices G and E because M_{GE} is between G and E, and it does not pass through the other vertex, M_{FG}, because M_{FG} is not F. Thus, the conditions of Pasch's axiom are met and $\overleftrightarrow{FM_{GE}}$ must intersect either $\overline{EM_{FG}}$ or $\overline{M_{FG}G}$. Since it intersects $\overleftrightarrow{M_{FG}G}$ at F, it cannot also intersect this line in the segment $\overline{M_{FG}G}$. Thus, it intersects $\overline{EM_{FG}}$. This proves that the medial point of a triangle is always inside the triangle.

Analogously, the incenter of a triangle is always inside the triangle. The only difference in the proof is that, instead of knowing that the bisectors of vertex angles E and F intersect the opposite sides at M_{FG} and M_{GE}, respectively, we must first invoke the crossbar theorem to prove that they intersect the opposite sides *somewhere* on them, and give these points labels; say, E^* and F^*.

By the triangle postulate, three noncollinear points fully define a triangle and, since the medial point and the incenter have now been proven to be inside the triangle, they are fully defined. Because we nowhere invoked the parallel postulate, medial points and incenters always exist in **neutral geometry** and are thus topics of discussion for white and yellow belts. But what about the orthocenter? A triangle's apex altitude is inside it only if the base angles are **acute**, so white and yellow belts may only discuss the orthocenter if the triangle is known to be acute. By a somewhat more involved argument, the circumcenter also exists for acute triangles. Sometimes these centers exist for triangles that are slightly **obtuse**, though giving a precise meaning to "slightly obtuse" is beyond the scope of this book; thus, white and yellow belts are advised to just defer most discussions of these triangle centers to orange belt.

[16] Midpoints are defined in Construction 1.2, and their existence inside the segment assured. Triangle centers will be defined later, and their existence assured. Here we speak casually of things that will be treated rigorously later.

This concludes our discussion of the postulates of geometry. But you may still be wondering, what is geometry about? The first line of a book is often the only thing people remember about it.[17] Like Herman Melville, Euclid is also famous for his first line – but not in a good way. "A point is that which has no part." Beginning geometry students are like, "Oh, so this is a book about Japanese koans?" In the first paragraph of *Geometry–Do*, there are a dozen boldface terms for the student to look up in the glossary, so you may be thinking, "Oh, so this is a book about memorizing vocabulary? It is like learning a language spoken in a country that I will never visit?"

When I was a freshman in college, I rather inadvisably took an upper-division course on groups, rings and fields. Why not? If not with knowledge, I was at least filled with ambition! Most of the material in this introduction came from that textbook, but what I remember most is the first line:

The main business of mathematics is proving theorems.

John Fraleigh (1989) set this sentence between horizontal lines, just as I have done above. He must have thought it important! He is right; the business of mathematicians is proving theorems.

Example Theorem
The sum of quadrilateral ***diagonals*** *exceeds the sum of either pair of* ***opposite*** *sides.*

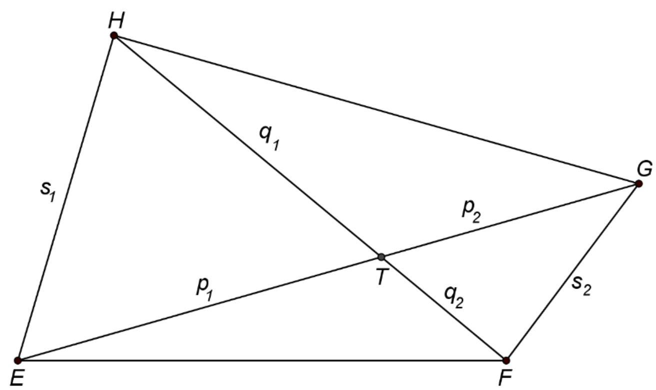
Let us try proving a theorem, and do so now, before the add/drop date, while there is still time for students to make a run for it! This will be fun. We will do it step by step, so I can lead the reader by the hand through a genuine geometry proof. It is easy, and it requires only the basics.

Step One: The first step is to remind yourself of the definitions of terms that you already know, and to look up any terms that you have not yet learned. We read about the quadrilateral earlier; but let us look it up to make sure that we know it. Look up adjacent, convex, and diagonal too!

Quadrilateral	The union of two triangles adjacent on a side such that it is convex; $\overline{EFGH}$
Adjacent	Two disjoint triangles with a common side (common for its full length)
Convex	Any segment between two points interior to two sides is inside the figure
Diagonal	Segments connecting non-consecutive quadrilateral vertices

[17] For instance, "Call me Ishmael" is the first line of *Moby Dick*. I think it is about a whale – I don't remember.

Step Two: The next step is to draw the figure, and to do so in a way that the definitions of terms are satisfied. For instance, if our two triangles are like the blades on an arrowhead, then the figure is not convex; a segment between the trailing edges of the two blades would not be inside the figure. This is not a quadrilateral; the proof below does not work for it because its diagonals do not intersect. In the figure below, we see that $\overline{EG}$, the side common to the two triangles, $\overline{EFG}$ and $\overline{GHE}$, is a diagonal; indeed, because it defines the quadrilateral, it is called the **definitional diagonal**. The same quadrilateral can be defined two ways, with two different pairs of adjacent triangles. Sometimes it matters which diagonal is definitional, but the problem at hand mentions both diagonals, so draw both, $\overline{EG}$ and $\overline{FH}$, and label the cut segment lengths p_1, p_2, q_1, q_2. The intersection of the diagonals is labeled T, and two of the side lengths are labeled s_1 and s_2. Lowercase letters denote lengths and can be added; they are not a symbol for the segment itself.



Step Three. The next step is to go through the index and look for relevant postulates, axioms, and theorems. The index is fifty pages long; so, later, this can be a daunting task. Intuition and experience when carrying out this search is what divides passing green- and red-belt geometers from failing ones. But, at this early stage in your career, the index for the introduction amounts to only two pages, so it is not a long search. The problem is about comparing two sums, so let us remind ourselves about additive groups. By segment addition, the two diagonals are $p_1 + p_2$ and $q_1 + q_2$. Their sum is $(p_1 + p_2) + (q_1 + q_2)$. We are comparing two magnitudes; so, relevant is a theorem about one magnitude being less than another. It is the triangle inequality theorem!

Step Four: The final step is to carry out the proof. There are two triangles with their sides labeled. In $\overline{ETH}$, by the triangle inequality theorem, $s_1 < p_1 + q_1$. In $\overline{FTG}$, by the triangle inequality theorem, $s_2 < p_2 + q_2$. Add the two inequalities together: $s_1 + s_2 < (p_1 + p_2) + (q_1 + q_2)$. We do not have to go through this for the other pair of opposite sides; just say, “analogously.”

Thus, you have seen the business of mathematicians. If you have not concluded that being a waiter with a psychology degree is the life for you, then I will see you tomorrow! We will prove the dreaded side–angle–side (SAS) theorem, which stumped both Euclid and David Hilbert!

Experts only!!! (Those uninterested in Hilbert's *Foundations* skip to the notation section.)

Hilbert's "straight line" is redundant; there is no such thing as an unstraight line, so we will just say "line." Hilbert uses the letters $A, B, C, \ldots$ for points, but these will be changed to $E, F, G, \ldots$ to be compatible with *Geometry–Do*, where the first four letters have special meanings. "Always completely determine" is the same thing as "fully define," so we will use the *Geometry–Do* term. "Situated in the same line" means collinear; indeed, "situated on a line" can just be "on a line." "Passes through a point of the segment" means "intersect." I am trying to make this easy!

One reason why beginners are uncomfortable with Hilbert's axioms is their verbosity; sadly, this has often resulted in any mention of foundations being delayed until the students have become advanced. But going back and filling in foundations later is not the right way to teach geometry. This verbosity is largely because German is difficult to translate into English. Economists who have read Carl Menger and philosophers who have read Friedrich Nietzsche have also noticed this. The solution is to not translate quite so literally, which is what I have done below.

I. Axioms of Connection

1. Two distinct points fully define a line.
2. Any two distinct points of a line fully define it.
3. Three points not collinear fully define a plane.
4. Any three points of a plane not collinear fully define the plane.
5. If two points of a line are in a plane, then every point of the line is in the plane.
6. If two planes have a common point, then they have at least one other common point.
7. Lines have at least two points, planes at least three noncollinear points, and space at least four noncoplanar points.

II. Axioms of Order

1. If E, F, G are collinear and F is between E and G, then F is also between G and E.
2. If E and G are two points on a line, then there exists at least one point F that is between E and G and at least one point H so situated that G is between E and H.
3. Of any three collinear points, there is exactly one between the other two.
4. Any four collinear points E, F, G, H can always be arranged so F is between E and G; also, between E and H. Furthermore, G is between E and H; also, between F and H.
5. If E, F, G are not collinear and a line in the plane they determine does not pass through any of them and it intersects $\overline{EF}$, then it will also intersect either $\overline{EG}$ or $\overline{FG}$.

Some jokers have noticed that Euclid never said there had to be more than one point, and so they defined their own geometry with *exactly* one point. Every segment is of zero length and is at the same point; analogously, every circle has zero radius and has the same center. They took navel gazing to a whole new level, just staring at that one point, and seeing how many geometry theorems are true about it! This must have been meant as a joke, but I think Hilbert was a little too concerned about excluding these degenerate geometry theories and got a bit pedantic doing so with his axioms. *Geometry–Do* is comfortable leaving terms like plane and point undefined and assuming that the students are not going to play any jokes by twisting their meanings; they are just trying to learn some geometry that will be useful in their everyday lives. Thus, I. 7 is not made explicit. II. 1 – 3 are the glossary definition of "between" except that II. 1 has "between" assume collinearity rather than imply it. II. 4 is redundant and is omitted in *Geometry–Do*. II. 5 is Pasch's Axiom, which I include among the secondary axioms at the end of the introduction.

I. 5, 6 are omitted because *Geometry–Do* does not include solid geometry. Two reasons:

1. High-school students have enough on their plates with plane geometry. *Geometry–Do* is a three-year course, assuming I can squeeze blue belt, *Cho–Dan* and *Yi–Dan* into a single year. It is possible, especially if *Sam–Dan* is included, that this will be a four-year course.

2. Traditional solid geometry (*e.g.,* Wentworth or Kiselev) is not very useful. This material is better taught as an application of Calculus III. Wolfe and Phelps have an advanced version of *Practical Shop Mathematics* that is about solid geometry, but it does not cite any of Wentworth's theorems; civilian machinists are just not that into cones and spheres.

Indeed, while Wentworth is harmless, teaching teenagers too much about machining cones and paraboloids is risky because of their use in shaped charges and explosively formed projectiles. In high school, the volume formulas are just food for memorization. In Calculus III, these formulas can be derived, and the students are mature enough not to do anything crazy like making an EFP.

Geometry–Do postulates comparable to Hilbert's *Axioms of Connection and Order*

Segment	Two points fully define the segment between them.
Line	By extending it, a segment fully defines a line.
Triangle	Three noncollinear points fully define a triangle.

We are left with I. 1, 2, 3, 4, which are comparable to my segment, triangle, and line postulates. Hilbert's axioms are about points and lines; he defines segment almost as an afterthought. But *Geometry–Do* follows Euclid by having distinct segment and line postulates. This is wise because segments are foundational; they should not just be tossed in later. Also, I. 1, 2 are redundant; two points define a unique line, and a line is defined by any two points on it is just one postulate. I. 3, 4 is also just one postulate; let us compare it to the triangle postulate of *Geometry–Do*.

Hilbert is overreaching when he states that three noncollinear points fully define a plane. The Euclidean plane and the Lobachevskian plane are different things. Without a parallel postulate, existence of triangle centers is only assured inside the triangle with these vertices. To say, "the plane" requires explanation of what, exactly, has been defined. In the introduction, I write:

> *By the triangle postulate, three noncollinear points fully define a triangle and, since the medial point and the incenter have now been proven to be inside the triangle, they are fully defined. Because we nowhere invoked the parallel postulate in the preceding proofs, medial points and incenters always exist in neutral geometry… But what about the orthocenter? A triangle's apex altitude is inside it only if the base angles are acute, so white and yellow belts may only discuss the orthocenter if the triangle is known to be acute. By a somewhat more involved argument, the circumcenter also exists for acute triangles. Sometimes these centers exist for triangles that are slightly obtuse, though giving a precise meaning to "slightly obtuse" is beyond the scope of this book.*

III Axiom of Parallels

In a plane there can be drawn through any point not on a line, one and only one line that does not intersect the given line. This line is called the line's parallel through that point.

Hilbert's third group of axioms consists of only one axiom, which is the same as in *Geometry–Do*.

IV Axioms of Congruence

1. If E and F are two points on a line and J is a point on the same or another line, then, on a given side of J on this line, there exists a unique point K such that $\overline{EF}$ is congruent to $\overline{JK}$, which is written $\overline{EF} \equiv \overline{JK}$. Every segment is congruent to itself; $\overline{EF} \equiv \overline{EF}$.

I am not an historian; but, as far as I know, this is the first time anyone ever used the term congruent to mean that two segments are the same length. Euclid would have said that they are equal and, as evidenced by Kiselev and Wentworth, this continued to be the practice through the 19th century in both the East and the West. Equal refers to magnitudes because they are fully defined by a single measurement, *e.g.,* the length of a segment. Triangles have three sides and three angles but – only after proving some theorems – we know that it is possible to measure three magnitudes and have equality for all six. Congruence is not just a single measurement.

Also, Hilbert's notation is confusing, though this may be due to the typesetting of his day. He did not use overlines while we use $\overline{EF}, \overrightarrow{EF}$ and $\overleftrightarrow{EF}$ to mean segment, ray, and line, respectively. He used only $\equiv$ while we use $=, \cong$ and $\equiv$ to mean equals, congruent and coincident, respectively.

IV Axioms of Congruence

1. Given $\overline{EF}$ and $\overrightarrow{JK}$, there exists a unique point L on $\overrightarrow{JK}$ such that $\overline{EF} = \overline{JL}$.
2. If $\overline{EF} = \overline{JK}$ and $\overline{EF} = \overline{LM}$, then $\overline{JK} = \overline{LM}$.
3. F is between E, G; also, K is between J, L. If $\overline{EF} = \overline{JK}$ and $\overline{FG} = \overline{KL}$, then $\overline{EG} = \overline{JL}$.
4. Given $\angle EFG$ and $\overrightarrow{KJ}$, there exist rays $\overrightarrow{KL}$ and $\overrightarrow{KL''}$ such that $\angle EFG = \angle JKL = \angle JKL''$.
5. If $\angle EFG = \angle JKL$ and $\angle EFG = \angle MNO$, then $\angle JKL = \angle MNO$.
6. If $\angle EFG = \angle JKL$ and $\overline{EF} = \overline{JK}$ and $\overline{FG} = \overline{KL}$, then $\angle FGE = \angle KLJ$ and $\angle GEF = \angle LJK$.

Hilbert's axioms of congruence are here written using *Geometry–Do* notation. I wrote all six of them in six lines, while Hilbert uses a total of 33 lines. This verbosity is one reason why Hilbert is no longer taught to beginners, though this does not justify omitting any discussion of the axiomatic method or just giving it lip service, as is typical these days.

Hilbert is not saying much here. IV. 1, 4 are the replication axiom, IV. 2, 5 are transitivity, and IV. 3 is substitution of equals in addition, which apparently applies only to segments, but not to angles. Frankly, my statement that "a set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a magnitude" and that there are three geometric magnitudes – lengths, angles, and areas – is a *lot* clearer and more succinct. Also, it is more complete. Why does IV. 3 not have an analogous statement about angles? What about area? Why is only transitivity mentioned and not the reflexive, symmetric and anti-symmetric relations? This is a *very* sketchy description of the properties of equivalence relations, total orderings, and additive groups.

IV. 6 is SAS congruence, though Hilbert makes $\overline{EG} = \overline{JL}$ a theorem. I prove SAS by citing the triangle postulate, which Hilbert could not do because he said that three noncollinear points fully define the plane, which is not what is needed to prove SAS. Hilbert is defining the plane to distinguish it from other planes in the context of solid geometry; we just want to prove SAS.

There is no axiom comparable to my circle postulate; Hilbert just inserts the definition of circle immediately before moving on to Archimedes' axiom, which is a bad idea for the same reason that casually inserting the definition of segment is. Segments and circles are foundational and deserve their own postulates.

Geometry–Do also has Archimedes' axiom; it is among the secondary axioms in the introduction.

Straight angles equal each other if and only if right angles equal each other. Straight is undefined and we could say that it is intuitive that they are all equal, as Hilbert does, or we could use Euclid's postulate, as I do. It is the same thing; Hilbert is too hard on Euclid when he calls him wrong. My right-angle postulate is, "all right angles are equal; equivalently, all straight angles are equal."

Notation

$\alpha,\ \beta,\ \gamma,\ \delta$	Angles of a triangle or quadrilateral; usually $\angle E, \angle F, \angle G, \angle H$, respectively. If α and β are base angles of a triangle, then $\delta = \|\alpha - \beta\|$, the skew angle.
$\rho,\ \sigma,\ \varphi$	ρ is right, σ is straight, and φ is the interior angle in an equilateral triangle.
$E,\ F,\ G, \ldots, W$	Points. H, I, O, R, S, T, U, V have assigned meanings; do not use arbitrarily.
M, I, X, Y, Z	M is usually inside a segment; M_{EF} is the midpoint of $\overline{EF}$. Otherwise, double subscripts denote reflection. I is the incenter, X, Y, Z are the excenters and, when subscripted with E, F, G, their pedal points.
$E',\ F',\ G'$	The feet of perpendiculars from E, F, G, particularly the altitudes of $\overline{EFG}$
$E^*,\ F^*,\ G^*$	Infeet; intersections of angle bisectors with the opposite sides of a triangle
$e,\ f,\ g$	Lengths of the sides of a triangle opposite the E, F, G vertices, respectively
$a,\ b,\ c$	The coefficients of $ax^2 + bx + c = 0$; use u, v, w for right triangles.
H, h, h_E, h_F, h_G	H is usually a triangle's orthocenter unless it is the fourth vertex of a quadrilateral. h is the height of a triangle or parallelogram if given a base. h_E, h_F, h_G are the altitudes dropped from $E,\ F,\ G$.
A, B, C, D, d	A is the area of a triangle or quadrilateral, *e.g.,* $\overline{EFG}$ has area $A = \left\|\overline{EFG}\right\|$; B is a solid's base area; C is a triangle's medial point or a parallelogram's bi-medial point; D is the circumdiameter; and d is the indiameter.
P	A point, usually interior. P_E, P_F, P_G are the pedal vertices of P in $\overline{EFG}$.
L_E, L_F, L_G	Long centers of $\overline{EFG}$, where the mediators and angle bisectors meet on ω
$r,\ R$	R is circumradius; r is inradius or other radii if there is no incircle present.
s, S, T, U, V, ℓ	s is the semiperimeter; S is the anticenter if T is not; and T is the bi-medial. U and V are the first and second Torricelli points. ℓ is a labeled line.
ω, O	ω (omega) is a circle, usually the circumcircle; O is usually a circle's center
$\equiv,\ \cap,\ \cup,\ -,\ \in, :=$	Coincident, intersection, union, removal, element of a set, assign to a label
$\perp,\ \parallel,\ \nparallel,\ \cong,\ \ncong\ \sim$	Perpendicular, parallel, not parallel, congruent, not congruent, and similar
$\|P\|, \left\|\overline{EF}\right\|, \left\|\overline{EFG}\right\|, \left\|\overline{EFGH}\right\|, \|x - y\|$	Power of a point, unit length, area, area, absolute value

White Belt Instruction: Foundations

Side–Angle–Side (SAS) Theorem ***(Euclid, Book I, Prop. 4)***
*Given two **sides** and the angle θ between them, $0 < \theta < \sigma$, a triangle is fully defined.*

> ***Proof***
> By the segment postulate, the segments have two endpoints and, since they form an angle $0 < \theta < \sigma$, they share an endpoint. This is three noncollinear points so, by the triangle postulate, the triangle is fully defined. Congruence is transitive, so any two anywhere are congruent. ∎

Euclid had five postulates, not six, but proof of his fourth proposition, SAS congruence, relied on superposition, which tacitly assumes a whole slew of additional and unmentioned postulates. Many have cast doubt on Euclid, pointing out that superposition – sliding figures around and flipping them over to position one on top of the other – is nowhere defined.

Robin Hartshorne (2000, p. 2), writes, "Upon closer reading, we find that Euclid does not adhere to the strict axiomatic method as closely as one might hope… The method of superposition… cannot be justified from the axioms… we can develop geometry according to modern standards of rigor." But, when *Common Core* was formulated, Hartshorne was shunted aside because Bill Gates was offering big money to redefine congruence in terms of transpositions – sliding figures around on a computer screen to superimpose them – assuring that geometry ceases to exist the moment a student rises from his school computer. By this definition, is a $3 : 4 : 5$ triangle drawn in this book congruent to one drawn on the wall of a 4000-year-old pyramid in Egypt? Neither moved! For that matter, did a figure in this book fly through the air and land on your homework?

Isosceles Triangle Theorem ***(Euclid, Book I, Prop. 5)***
If two sides of a triangle are equal, then their opposite angles are equal.

> ***Proof***
> Given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, by SAS, $\overline{FGE} \cong \overline{EGF}$ because $\overline{FG} = \overline{EG}$ and $\angle FGE = \angle EGF$ and $\overline{GE} = \overline{GF}$. By congruence, $\angle EFG = \angle FEG$. ∎

Observe that, when we cite SAS, the triangle vertices are ordered by the side, angle and side that are equal; later, in more advanced proofs, we will not write "because" and list the equalities.

$\overline{FGE}$ and $\overline{EGF}$ have the same vertices but they are different triangles. $\overline{FGE} \cong \overline{EGF}$ is not a trivial statement proven by reflexivity; it requires proof, and it has important implications. The triangle postulate states that three noncollinear points fully define a triangle, but only in the order given.

Equilateral Triangle Theorem
*Given a triangle, the following are equivalent: (1) It is **equilateral**; (2) all interior angles are equal; (3) the **medians**, the **altitudes**, and the angle **bisectors** are pairwise coincident; (4) the three medians are equal; (5) the three altitudes are equal; (6) the three angle bisectors are equal.*

Half Equilateral Triangle Theorem
*A triangle is **half equilateral** if and only if it is right and one **leg** is half of the **hypotenuse**.*

Proof of the SSS theorem will use a proof by **contradiction**; that is, show that q not true and p true is contradictory. We have defined dichotomy and trichotomy; now we assume that G and J are distinct and then consider the four places where J can be if it is not G. Like aiming a rifle at a target, there are only five alternatives: a bullseye or a miss to the left, right, above, or below. We show that the latter four are impossible. The **lemma** is based on what "inside" means.

Lemma 1.1
*If a triangle is **inside** another triangle, it has less area.*

Side–Side–Side (SSS) Theorem ***(Euclid, Book I, Prop. 8)***
Given three sides that satisfy the triangle inequality theorem, a triangle is fully defined.

Proof
Given $\overline{EFG}$ and $\overline{EFJ}$ with $\overline{EG} = \overline{EJ}$ and $\overline{FG} = \overline{FJ}$, suppose that G and J are distinct. By lemma 1.1, if J is inside $\overline{EFG}$ or inside the angle **vertical** to $\angle EGF$, then $|\overline{EFJ}| < |\overline{EFG}|$ or $|\overline{EFJ}| > |\overline{EFG}|$, respectively, which implies $\overline{EFG} \ncong \overline{EFJ}$. Suppose J is on the E side of $\overleftrightarrow{FG}$ but not inside $\overline{EFG}$. $\overline{EG} = \overline{EJ}$, so $\overline{EGJ}$ is isosceles. $\angle EJG = \angle EGJ$ by the isosceles triangle theorem. By analogous reasoning, $\overline{FGJ}$ is isosceles and thus $\angle FGJ = \angle FJG$.

$\angle EJG = \angle FJG \cup \angle EJF$	and by analogous reasoning	$\angle FGJ = \angle EGJ \cup \angle FGE$
$\angle EJG > \angle FJG$		$\angle FGJ > \angle EGJ$
$\angle EJG > \angle FGJ$		$\angle FGJ > \angle EJG$

A contradiction; J on the F side of $\overleftrightarrow{EG}$ but not inside $\overline{EFG}$ is also contradictory. ∎

In the following constructions, rays and lines are announced without invoking the line postulate; this is in keeping with our plan to avoid tedious proofs with mincing steps. By construction, midpoints, angle bisectors and perpendiculars to a line through a point are fully defined. Metric geometry textbooks begin with the midpoint theorem – every segment has exactly one midpoint – which they prove by dividing by two. But they never explain how a real number was assigned to the length or, after division, how to locate the midpoint. It just appears!

Construction 1.1 *Bisect an angle.* ***(Euclid, Book I, Prop. 9)***

Solution

Given $\angle EFG$, take any point J on $\overrightarrow{FE}$. There exists a point K on $\overrightarrow{FG}$ such that $\overline{FJ} = \overline{FK}$. Construct an isosceles triangle with **base** $\overline{JK}$ and **apex** L on the other side of $\overleftrightarrow{JK}$ from F. By SSS, $\overline{JFL} \cong \overline{KFL}$, which holds the equality $\angle JFL = \angle KFL$. ■

To construct an isosceles triangle when the base is given, a geometer sets his compass to any length longer than half the base and draws arcs from each endpoint. Where these arcs intersect is an apex; there are two possible, one on each side of the base. These arcs are each called a **locus**, and together, **loci** (lō' sī). To construct an isosceles triangle when the apex angle is given, lay off the same arbitrary length on both rays from the **vertex** and then connect these points.

Construction 1.2 *Bisect a segment.* ***(Euclid, Book I, Prop. 10)***

Solution

Given $\overline{EF}$, construct an isosceles triangle with $\overline{EF}$ the base and G the apex angle. Using C. 1.1, bisect the apex angle, $\angle EGF$. (When finding G, swing your compass around to find G'' on the other side of $\overleftrightarrow{EF}$.) Let $\overrightarrow{GG''}$ cut $\overline{EF}$ at M.[18] By SAS, $\overline{EGM} \cong \overline{FGM}$, which holds the equality $\overline{EM} = \overline{FM}$; that is, M is the midpoint of $\overline{EF}$, so $M \equiv M_{EF}$. ■

Construction 1.3 *Raise a perpendicular from a point on a line.* ***(Euclid, Book I, Prop. 11)***

Solution

Given a line with M on it, lay off the same arbitrary length to the left and to the right of M, so $\overline{EM} = \overline{FM}$. Construct an isosceles triangle with base $\overline{EF}$ and apex G. By SSS, $\overline{EMG} \cong \overline{FMG}$, which holds the equality $\angle EMG = \angle FMG$, so these are right angles. ■

Construction 1.4 *Drop a perpendicular from a point to a line.* ***(Euclid, Book I, Prop. 12)***

Solution

Given G not on $\overleftrightarrow{EF}$, construct an isosceles triangle with apex G and base $\overline{JK}$ on $\overleftrightarrow{EF}$. The apex angle bisector, $\overrightarrow{GG''}$ (construct it in the same way as in C. 1.2) cuts $\overline{JK}$ at M. By SAS, $\overline{JGM} \cong \overline{KGM}$, which holds the equality $\angle JMG = \angle KMG$, so these are right angles. ■

These constructions are the four basic techniques that will be used in combination throughout geometry. At the most fundamental level, all four are much alike. This is analogous to how the

[18] Moise (1990, p. 83) derides the "lighthearted use of the word *let*." Not us! We *proved* the crossbar theorem!

jab, hook, uppercut, and cross are the basic techniques that are used in combination throughout boxing. But all four involve giving somebody a poke in the nose, so they are much alike. Did you get the equilateral triangle theorem? You only had two theorems in your kit! Like a carpenter who only owns a claw hammer, for every nail, he is either going to hit it or pry it out. What else?

Construction 1.5 *Replicate an angle.* ***(Euclid, Book I, Prop. 23)***

Solution

Construct an isosceles triangle with the given angle as its apex angle by laying off equal lengths and connecting them. By SSS, reconstruct this triangle elsewhere. ■

Construction 1.6 *Given a ray and a point on the angle bisector, find the other ray of the angle.*

Solution

Given $\overrightarrow{EF}$ and P on the angle bisector, construct an isosceles triangle with apex E and base $\overline{PJ}$ with J on $\overrightarrow{EF}$ so $\overrightarrow{EJ} \equiv \overrightarrow{EF}$. By C. 1.5, construct $\angle KEP$ equal to $\angle JEP$ by using SSS to construct $\overline{KEP} \cong \overline{JEP}$ with J and K on **opposite** sides of $\overleftrightarrow{EP}$. $\overrightarrow{EP}$ bisects $\angle JEK$. ■

The perpendicular bisector of a segment is called its **mediator**. The perpendicular from a triangle vertex to the (extension of the) opposite side is the altitude. Altitudes and angle bisectors can be extended past the opposite side, but when lengths are assigned to an altitude or to an angle bisector, it means the length of the segment from the vertex to the opposite side.

Center Line Theorem

An angle bisector and a perpendicular bisector coincide if and only if the triangle is isosceles.

Proof

Assume the angle bisector and perpendicular bisector coincide. By SAS (segment reflexivity, the right-angle postulate and segment bisection), the two **right triangles** are congruent, so their hypotenuses are equal. Thus, the given triangle is isosceles. •

Assume the triangle is isosceles. By the isosceles triangle theorem, the base angles are equal. Construct a median from the apex. By SAS (opposite sides, opposite angles, and bisection), the two triangles are congruent. The apex angle is bisected and the angles at the **foot** of the median are equal; both right because they bisect a straight angle. • ■

The **center line** is the mediator of the base and the apex angle bisector of an isosceles triangle.

The center line theorem is **bi-conditional** and so it requires two independent proofs, concluded with •. The mediator theorem will also be like this. The two proofs may be done in either order.

Technically, p and q are equivalent even if proof that q implies p requires citing the previously proven statement that p implies q. However, students see it as a trick if I say, "prove that p and q are equivalent," but I do not mention that they must prove that p implies q first, and *then* prove that q implies p. No tricks! If this is the case, then I will call the statement that p implies q a theorem, and the statement that q implies p its converse, but I will not call them equivalent.

Interior and Exterior Angles Theorem
The bisectors of an interior and ***exterior*** *angle of a triangle are perpendicular to each other.*

> ***Proof***
> Given $\overline{EFG}$ and J on $\overrightarrow{EF}$ past F, $\angle EFG$ is the interior angle and $\angle JFG$ is the exterior angle at vertex F. By C. 1.1, find K and L on the angle bisectors of $\angle EFG$ and $\angle JFG$, respectively. $\angle EFK = \angle GFK$ and $\angle JFL = \angle GFL$, so $\angle EFK \cup \angle JFL = \angle GFK \cup \angle GFL$. The union of these four angles is a straight angle and, if a straight angle is cut in two equal angles, then each one is right; thus, $\angle GFK \cup \angle GFL = \rho$ and $\overrightarrow{FK} \perp \overrightarrow{FL}$. ■

Mediator Theorem
A point is on the perpendicular bisector iff it is ***equidistant*** *from the endpoints of the segment.*

> ***Proof***
> Assume that G is on the perpendicular bisector of $\overline{EF}$, but it is not M_{EF} (if it is, then we are done). By SAS, $\overline{EM_{EF}G} \cong \overline{FM_{EF}G}$, which holds the equality $\overline{GE} = \overline{GF}$. •
>
> Assume $\overline{GE} = \overline{GF}$. Connect $\overline{GM_{EF}}$. By SSS, $\overline{EGM_{EF}} \cong \overline{FGM_{EF}}$, which holds the equality $\angle EGM_{EF} = \angle FGM_{EF}$. Thus, $\overrightarrow{GM_{EF}}$ is the angle bisector of $\angle EGF$ and, by the center line theorem, it is the perpendicular bisector of $\overline{EF}$. • ■

Problem 1.1 *Draw a line through a point so it cuts off equal segments from the rays of an angle.*

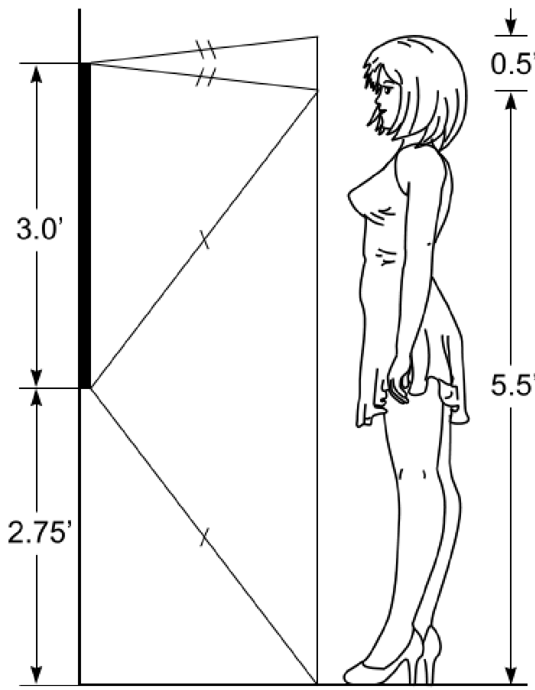
> ***Solution***
> By the definition of isosceles, the desired line is the base of an isosceles triangle with the given angle at its apex. By the center line theorem, the base is perpendicular to the apex angle bisector. Bisect the angle and drop a perpendicular on it from the point. ■

Just solving a problem is not enough; you must also explain in what situations your solution might fail. One can always drop a perpendicular on a line, but not always on a ray, so this may not work. Sometimes there are two or more solutions to a problem, and you must explain why and under what conditions the number of solutions changes. This is called the **discussion**.

Problem 1.2 *A fink truss consists of an equilateral triangle built on the middle third of the ceiling joists. The rafters rest on the walls and meet at the triangle apex. Beams from the feet of the triangle meet the rafters at right angles. Draw it. The boards need not have width.*

Here, the roof's slope is $\frac{\varphi}{2}$; steep, but very strong. The king post and queen post trusses are more versatile, handling arbitrary and flatter slopes. Look them up if you are interested.

Problem 1.3 *Suppose your girlfriend asks you for a wall mirror. She is six feet tall in heels and her eyes are six inches below the top of her hair. What is the smallest mirror that allows her to see her entire self and how high should it be above the floor? Does it matter how far away she stands?*



Construct two isosceles triangles with bases from her eyes to her feet and to the top of her hair.

A carpenter constructs an A-frame with E and F the feet, G the apex, $\overline{EG} = \overline{FG}$ and a crosspiece between M_{GE} and M_{FG}, just like a commercial steel A-frame. But, when it is overloaded, the legs bow outward and start to pull free of the crosspiece. Reasoning that wood can take a compressive load but cannot pull things together while steel is just the opposite, he determines to connect $\overline{EM_{FG}}$ and $\overline{FM_{GE}}$ with wire rope to pull the bowed legs in tight with the crosspiece.

Problem 1.4 *Suppose that you are the carpenter who built the A-frame described above.*

1. *There are two different ways to prove that $\overline{EM_{FG}} = \overline{FM_{GE}}$. Prove this both ways.*

2. *Another carpenter criticizes your design, stating that, if the wire ropes are both attached to an anchor hammered into the ground at M_{EF}, he can prove that the two wires are equal and, thus, that this is the best design. How do you respond?*

3. *You wish to build the strongest possible A-frame with the given boards and believe that this is accomplished by having the wire rope pull on the legs perpendicular to the bow in the boards. Prove that this is true if and only if $\overline{EFG}$ is equilateral.*

4. *Construct an A-frame with wire ropes from each foot to the trisection points of the opposite legs and with the bottom wire ropes meeting the opposite legs at right angles.*[19] *Constructing this is too difficult; just draw it with a base of 13 cm and legs of 15.9 cm.*

White Belt Exit Exam

Saccheri Theorem I

*If $\overline{EFGH}$ is a **Saccheri quadrilateral**, so $\angle E = \angle F = \rho$ and $\overline{HE} = \overline{FG}$, then prove that*

1. $\overline{EG} = \overline{FH}$
2. $\angle G = \angle H$
3. $\overleftrightarrow{M_{EF}M_{GH}} \perp \overleftrightarrow{EF}$ *and* $\overleftrightarrow{M_{EF}M_{GH}} \perp \overleftrightarrow{GH}$
4. *The mediators of the base and the **summit** coincide.*

Rhombus Theorem

*Given a **rhombus** $\overline{EFGH}$, connect $\overline{FH}$. Without adding any **auxiliary** lines, prove that*

1. $\angle EFG = \angle GHE$
2. $\angle FGH = \angle HEF$
3. $\overline{FH}$ *bisects both* $\angle EFG$ *and* $\angle GHE$
4. *Draw the other diagonal, $\overline{EG}$, and prove that they are perpendicular bisectors.*

Isosceles Triangle Theorem Converse (White Belt)

If two angles of a triangle are equal, then their opposite sides are equal.

Perform these constructions:

1. *Construct a right triangle given one leg and the median from (a) that leg (b) the other leg.*
2. *Construct an isosceles right triangle so its apex altitude lies on a given line.*
3. *Construct an equilateral triangle so its apex altitude lies on a given line.*

---

[19] Connect the upper trisection points with a board so you can tighten the wire ropes against it. If you agree to call this the "Aguilar A-Frame," I will make it easy for you by just telling you that the base is 81.65% of the legs' length. Note that, because the solution is a ratio, this must be a blue-belt problem. Geometry with multiplication!

Practice Problems: *Construct each triangle using only the information given about it.*

1.5 *Construct a right triangle given the lengths of the legs.*

1.6 *Construct a triangle given the lengths of the three sides.*

1.7 *Construct a triangle given the apex angle and the lengths of the legs.*

1.8 *Construct a triangle given the lengths of the base, the median to the base and one leg.*

1.9 *Given* $\overline{EFGH}$, *if* $\overline{EF} = \overline{GH}$ *and* $\overline{FG} = \overline{HE}$, *prove that* $\overline{EFG} \cong \overline{GHE}$ *and* $\overline{FGH} \cong \overline{HEF}$.

1.10 *Given* $\overline{EFG}$ *with* $\overline{GE} = \overline{GF}$, $\overrightarrow{GE}$ *is extended to* E'' *and* $\overrightarrow{GF}$ *to* F''. *Prove* $\angle FEE'' = \angle EFF''$.

1.11 *Given* $\overline{EFG}$ *with* $\overline{GE} = \overline{GF}$, *construct an isosceles triangle,* $\overline{EFJ}$, *with the same base but not necessarily congruent to* $\overline{EFG}$. *Prove that* $\angle GEJ = \angle GFJ$.

1.12 *Given* $\overline{EFG}$ *with* $\overline{GE} = \overline{GF}$, *find points* J *and* K *on* $\overline{EF}$ *such that* $\overline{EJ} = \overline{FK}$. *Prove that* J *and* K *are also equidistant from the vertex; that is,* $\overline{GJ} = \overline{GK}$.

1.13 *The same as P. 1.12, but with* J *on* $\overrightarrow{FE}$ *past* E, *and* K *on* $\overrightarrow{EF}$ *past* F.

1.14 *Given* $\overline{EFG}$ *with* $\overline{GE} = \overline{GF}$, *prove that,*

1. $\overline{M_{FG}M_{GE}M_{EF}}$ *is isosceles.*
2. $\angle GM_{GE}M_{EF} = \angle GM_{FG}M_{EF}$
3. $\angle EM_{EF}M_{GE} = \angle FM_{EF}M_{FG}$

1.15 *Given two lines that intersect to make one right angle, prove that the others are also right.*

1.16 *Ancient hieroglyphics describe a* $350'$ *tall pyramid that had all the same dimensions as the Luxor hotel in Las Vegas, but it was reduced to rubble thousands of years ago. Could a* Common Core *student prove it congruent to the Luxor hotel by using superposition?*

1.17 *Your school has a foreign exchange student – from Mars! He accepts all our postulates except the parallel postulate. The symbol of his people is a* $13 : 14 : 15$ *triangle chiseled into a stone temple on Olympus Mons. He insists that it is not congruent to any Earthling triangle. By comparing rulers, you find that his unit of length is* 3.219 *cm. Can you draw a triangle and prove that it is congruent to his symbol? Can a* Common Core *student?*

Pisa Tree Problem

You bought a laser rangefinder! Yay! *But now, your geometry seon-saeng [teacher] has challenged you to measure the vertical height of the Pisa Tree in front of your school, so called because it leans like the Tower of Pisa. Because all the branches obscure your view, you cannot aim your laser straight up, so you take two measurements from either side: From an arbitrary point, you measure the distance to the treetop as* 17 *meters and, from* 32 *meters away and directly across from a point directly below the treetop (this is called its projection), you measure the distance to the treetop as* 22 *meters. What is the vertical height of the tree in meters?* Beware! *We have no assurance that our world is Euclidean and not hyperbolic, at any scale.*

In the following constructions, you are not allowed to use a protractor. They are inaccurate when the angle is extended to the size of a house. Also, the students were not asked to buy one and so most of them did not. It is unfair for some students to use equipment the others do not have.

1.18 *Construct an equilateral triangle,* $\overline{EFG}$. *In the* Notation *section, we define* φ *to be the interior angle of an equilateral triangle. Is this the same thing as defining it to be a third of a straight angle? Is* $\overline{M_{EF}M_{FG}M_{GE}}$ *equilateral in hyperbolic geometry, or only Euclidean? Can you prove that the interior angles of* $\overline{M_{EF}M_{FG}M_{GE}}$ *equal the interior angles of* $\overline{EFG}$*?*

1.19 *Given the hypotenuse, construct a half equilateral triangle. Is this a* $30–60–90$ *triangle?*

1.20 *Inscribe a square in a given square. Now inscribe a different square in the given square.*

1.21 *Draw a king post roof truss with a right apex. The boards need not have width.*

1.22 *You wish to have black metal water pipes laid vertically on your roof, so the sun may heat water that is pumped through them. The plumbing supply store sells* $45°$ *elbows and you wish to use them so your pipes bend over the apex of your roof and lay flat on both sides. Draw a king post roof truss with this apex angle. The boards need not have width.*

1.23 *The same as problem 1.22, but with* $22.5°$ *elbows. This is quite a flat roof, so it is like the king post roof truss, but with the addition of vertical boards from the rafter midpoints dropped onto the ceiling joists because the angled boards are at such a low angle that they do not fully support the midpoints of the rafters. The boards need not have width.*

1.24 *Draw a queen post roof truss with a right apex. In this design, lay off equal lengths from the apex onto the rafters, connect these points and drop perpendiculars onto the ceiling joist. It is not particularly strong for holding up a snow load, but it makes for a neat box shape in the attic that can be paneled as a room. The boards need not have width.*

Comparison with *Common Core* Geometry

Common Core teachers present the isosceles triangle theorem after showing students the button on *Geometer's Sketchpad* for bisecting a segment. They never demonstrate bisecting a segment with compass and straightedge; they rely heavily on that magical midpoint button.

***Common Core* Proof of the Isosceles Triangle Theorem (*Glencoe Geometry*, p. 286)**

ΔLMP with $\overline{LM} \cong \overline{LP}$	Given
Let N be the midpoint of $\overline{MP}$.	Every segment has exactly one midpoint.
Draw an auxiliary segment $\overline{LN}$.	Two points determine a line.
$\overline{MN} \cong \overline{PN}$	Midpoint Theorem
$\overline{LN} \cong \overline{LN}$	Reflexive Property of Congruence
$\overline{LM} \cong \overline{LP}$	Given
$\Delta LMN \cong \Delta LPN$	SSS
$\angle M \cong \angle P$	CPCTC 😊

This is more complicated than the *Geometry–Do* proof: given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, $\overline{FGE} \cong \overline{EGF}$ by SAS, so $\angle EFG = \angle FEG$. It requires an auxiliary line (bisecting a segment is 4 more steps, so 12 total), but it is easier because students need not understand that the same three points can define different triangles depending on how they are ordered. This is important![20] [21]

The Common Core proof requires SSS and thus cannot be used in *Geometry–Do* because the proof of SSS requires the isosceles triangle theorem. David Coleman dodges the charge of circular reasoning by the simple expedient of not proving SSS. For him, SAS and SSS are both postulates or, if called theorems, they are "proven" with tracing paper. *Cheater! Cheater! Booger eater!*

Common Core states the triangle similarity theorem as an axiom – we prove it in the blue belt chapter – calling it either the similarity axiom or the dilation axiom, and then state without proof the AA, SAS and SSS similarity theorems. SAS, SSS, ASA, AAS and HL are then just special cases of the similarity/dilation axiom with the scale (dilation factor) being the multiplicative identity – which requires assuming the field axioms for real numbers – and the mid-segment theorem is a special case with the scale (dilation factor) being one half. *Common Core* students who claim to know of easier proofs to the isosceles triangle theorem and its converse can only say this because they did not have to prove SAS, SSS, ASA, AAS and HL. *Common Core* is just boring memorization!

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[20] "Although a good proof of the theorem was known in antiquity, it has become customary in later centuries to prove it in needlessly complicated ways; and probably the worst of these rambling detours is the proof that starts by telling you to bisect [the apex angle] (Moise, 1990, p. 83)." C. 1.2 cites C. 1.1, so *Common Core* is this proof.

[21] Good job Grasshopper! You are still with us! Many got to page one and wailed, "He just renamed $\overline{FGE}$ as $\overline{EGF}$. It's the same triangle!" Then they dropped out. When you are an engineer, one of them will vacuum your office.

The orange-belt chapter concludes with a section on how to pass a standardized exam of the type that is designed for *Common Core* students. Most of the people now reading these lines will not survive orange belt, so I will here tell you how a *Geometry–Do* white belt can pass *Common Core* exams. First, recognize that it is really an algebra exam in disguise, so review Algebra I. But the big secret is to bring a center-finding metric ruler and a compass to the exam so you can construct the figures – the ones provided are purposefully wrong – and measure the unknown quantity.

Varsity Tutors Advanced Geometry Exam[22], problem #14, is solved below, first using geometry, and then using the algebra that masquerades as geometry in *Common Core*. Which is easier?

Problem 1.25 *If a triangle has base* 14 *cm and legs* 13 *cm and* 15 *cm, what is its apex height?*

Geometry Solution

Use SSS to construct the triangle and then measure its height. It is 12 cm! ■

Algebra Solution

Let x and y be projections of the 13 cm and 15 cm legs onto the base, respectively. Then $x + y = 14$ cm and, by the Pythagorean theorem, $13^2 = x^2 + h^2$ and $15^2 = y^2 + h^2$. Solve both equations for h^2, set them equal and substitute $y = 14 - x$ into the latter.

$169 - x^2 = 225 - (14 - x)^2$

$169 - x^2 = 225 - 196 + 28x - x^2$

$0 = -140 + 28x$

$x = \frac{140}{28} = 5$ cm

Substitute $x = 5$ into the first Pythagorean equation, $13^2 = x^2 + h^2$, then solve it for h.

$h = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$ cm ■

Varsity Tutors considers this advanced because almost no American geometry student can answer it correctly or, if they do, it takes them thirty minutes to work through all the algebra. But, if you construct the geometric figure with a ruler and compass (*Duh!* It is a geometry exam!), you can solve it in one minute using the most basic white-belt theorem you know.

Teachers! If you have read this far hoping for advice on how to get your #%$^@ students through the *Common Core* standardized exam, here it is: Ask for the perimeter of a triangle[23] with vertices $(-2,3), (-4,-4), (-7,-1)$ and make it a race. The easy way is to lay the three sides end-to-end on a line. Taking the sum of three applications of the algebraic distance formula is the hard way.

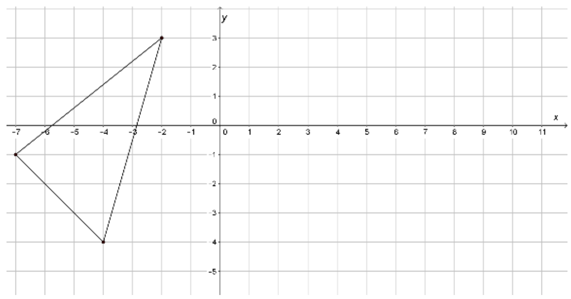
$$\sqrt{\left(-2-(-7)\right)^2+\left(3-(-1)\right)^2}+\sqrt{\left(-2-(-4)\right)^2+\left(3-(-4)\right)^2}+\sqrt{\left(-7-(-4)\right)^2+\left(-1-(-4)\right)^2}\approx 17.9$$

[22] *Varsity Tutors* practice exam: www.varsitytutors.com/advanced_geometry_diagnostic_1-problem-36916

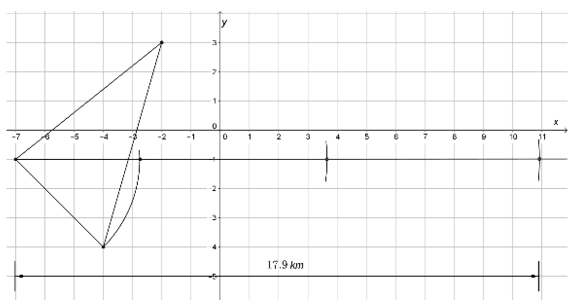
[23] A printout of the day-one exam: www.axiomaticeconomics.com/day_one_exam.pdf

First-Day Exam in Geometry

The first task of the high-school geometry teacher is to disabuse students of the notion that geometry is just a boring review of Algebra I. (Nothing new here. *Blah!!!*) You own a triangular pasture with vertices $(-2,3),(-4,-4),(-7,-1)$, as measured in kilometers. To the nearest 100 meters, how long is the fence around it? Make it a race with the *first* solver getting an A.



The easy way is to lay the three sides end-to-end on a line. Put the compass pin at $(-7,-1)$ and rotate it to lay off the lower left side on the horizontal. Without moving the pin, measure the upper left side and lay it off on the horizontal past the one you just did. Finally, measure the upper right side and lay it off on the horizontal past the one you just did. *It is segment addition!*



Taking the sum of three applications of the algebraic distance formula is the hard way to do this.

$$\sqrt{\left(-2-(-7)\right)^2+\left(3-(-1)\right)^2}+\sqrt{\left(-2-(-4)\right)^2+\left(3-(-4)\right)^2}+\sqrt{\left(-7-(-4)\right)^2+\left(-1-(-4)\right)^2}$$
$$=\sqrt{(5)^2+(4)^2}+\sqrt{(2)^2+(7)^2}+\sqrt{(-3)^2+(3)^2}$$
$$=\sqrt{25+16}+\sqrt{4+49}+\sqrt{9+9}\;=\;\sqrt{41}+\sqrt{53}+\sqrt{18}\;\approx\;6.40+7.28+4.24\;\approx\;17.9\text{ km}$$

The following are very easy geometry problems that will put *Common Core* graduates to shame:

Yellow Belt: Prove that the sum of the legs of a right triangle is less than twice the hypotenuse.

Orange Belt: Given a triangle with vertices $(0,0),\left(\frac{25}{3},0\right)$ and $(3,4)$, drop an altitude from the right vertex. What is the sum of the inradii of the three triangles thus formed?

White Belt Geometry for Construction Workers

Problem 1.26 *Rip a board into equal-width slats. (Three in this example.)*

> ***Solution***
> Because no carpenter has ever made it to orange belt, I will here present the two transversals theorem unproven: Parallel lines that equally cut one **transversal** equally cut any transversal. Traverse the edges twice with a ruler held so it is easy to divide; *e.g.*, trisect a $5.5"$ wide board by angling the ruler so $0"$ and $6"$ lie on the edges. Set the saw guide so two thirds of the kerf is towards the edge and one third is towards the center. ■

George Birkhoff's axioms are called metric because they assume the field axioms for real numbers; those of David Hilbert and this author are called intrinsic because they do not. Birkhoff is assuming tape measures longer than one's workspace that do not droop and protractors that measure angles to such precision that they can be projected across one's workspace and the opposite side of the triangle is as accurate as can be measured with one's tape. Carpenters have no means of measuring angles with such precision and their tapes are only $25'$ long. The **Egyptian triangle** can verify that an angle is right, but it does not create a right angle. Finding the corners of a rectangle can be frustrating for carpenters who know only this. It works only if the sides are rigid and reach across the entire workspace, so there is no extrapolation error. The only time I recommend that construction workers use the Egyptian triangle is if they build an $8'$ wall, nail it to the floor, measure $6'$ from it, and then have two men stretch a tape diagonally; when their tape measures $10'$, nail the wall to the ceiling joists. It is vertical!

Squaring a $16'$ cabin is easy (the diagonal is $22'\ 7.5''$), but a rectangle with sides longer than a tape measure requires Thales' diameter theorem. No construction worker has ever made it that far, so I will break my vow against using unproven theorems and just present a cook-book recipe. A string can be extended six times longer than a tape measure and, because it is light weight, it does not droop when stretched across these long distances. Because a rectangle may be several times longer than your tape measure, you will need two strings in addition to your tape. Use a spring scale to put uniform tension on the string, about one Newton (100 grams) per meter.

Squaring a foundation must be achieved with no auxiliary lines outside it. This is because it may be in a hole if it is for a basement, or it may be surrounded by trees or cliffs if a plot of land was cleared and graded for a house being built in a forest or cut into a hillside. To make the house face a road, give the front the same compass heading as the center line of the road. To make the house face south, stand at the SW corner and aim $90°$ minus magnetic declination off magnetic north; *e.g.,* in Los Angeles, aim for $78°$ east. Note that this is a Euclidean construction.

Problem 1.27 *Square a house's foundation before pouring the concrete floor.*

Solution

Mark the front segment, $\overline{EF}$, with two stakes measured with a tape and oriented with a compass to be parallel to a road or to the east-west line; do not neglect declination. Loop the end of string S_1 over the E-stake, stretch it across the front and tie it to the F-stake. Drive a stake, O, into the ground near the center, but slightly towards the front and slightly towards the F-stake. Loop the end of string S_2 over the O-stake, stretch it to the F-stake, pinch it and then swing this radius around the O-stake until the arc intersects $\overline{EF}$. Drive in a stake at this intersection, E_1. Do not lose your pinched-off length! Lift the S_1 string off the E-stake and loop it over the E_1-stake. Stretch it over and past the center stake, O; simultaneously, swing string S_2 around the O-stake to point in the opposite direction, away from E_1. Stretch both strings so they coincide (lie on top of each other) and drive a stake, G_1, in at the end of the length pinched-off on S_2. $\angle E_1FG_1$ is right by Thales' diameter theorem. Loop the end of string S_1 over the F-stake, stretch it into ray $\overrightarrow{FG_1}$ and drive a stake G on this ray past G_1 to where a tape measures the length of the side of the house. Pinch off this length, $\overline{FG}$, lift the S_1 string off the F-stake and loop it over the E-stake. Lift the S_2 string off the O-stake, loop it over the F-stake, pinch off the length $\overline{EF}$, then lift it off the F-stake and loop it over the G-stake. Stretch both strings out and where their pinched off lengths intersect, drive a stake, H. $\overline{EFGH}$ is a rectangle. ■

This leads directly to a rectangle while the Pythagorean theorem converse (if $u^2 + v^2 = w^2$, then the triangle with these sides is right) is hit and miss.[24] P. 1.26 and P. 1.27 are orange- and green-belt, respectively, but we must help the carpenters, and they almost never get that far.

Many come to *Geometry–Do* with prejudice against deductive logic. Now is the time to rid ourselves of these losers! They are baggage we will not need to bring to yellow-belt geometry. Put construction workers and others who come to geometry with an open mind on Team Euclid. Put those who have closed their minds to deductive logic and believe only in coordinate geometry on Team Prástaro. In two classrooms, push the desks to the walls, staple butcher paper to the ceiling and draw a chalk line on it. Give each team a yardstick, two spools of chalked string and two ladders. A team that can draw a chalk line on the floor directly underneath the one on the ceiling gets an A, else an F. They cannot use a plumb bob, but you will test their answers with it. If the losers on Team Prástaro demand a tape measure instead of a yardstick, explain that, unless you are building an outdoor toilet, rulers are always less than the length of one's workspace.

[24] To "X it" is to measure the four sides to construct a parallelogram and then adjust it until the diagonals are equal. Like the Pythagorean theorem converse, it can verify a right angle, but its failure does not tell you how to adjust.

The Egyptian or $3 : 4 : 5$ Right Triangle

In the preceding section I wrote, "Finding the corners of a rectangle can be frustrating for carpenters who know only this." So true! I remember when I was eight that my father had my mother, my brother and I at stakes marking three corners of the foundation of the basement for our house. He kept measuring sides one at a time with his only tape measure and ordering a stake moved a few inches this way or that. The Pythagorean equation never came out exact and it offered no hints on how to move the stakes to make it exact. Bad day!

In *Volume Two: Geometry with Multiplication*, the Pythagorean equation will be expressed as $u^2 + v^2 = w^2$ with u, v, w being real numbers. However, real numbers were only introduced in the 1800s and the modern theory of rational numbers did not precede them by much. Yet Egyptologists assure us that triangles with sides of $3, 4$ and 5 units appear in four-thousand-year-old hieroglyphics. We will do the ancient proof and, in *Volume Two*, we will do it rigorously.

Egyptian Triangle Theorem
A triangle with sides three, four and five times a unit length is right.

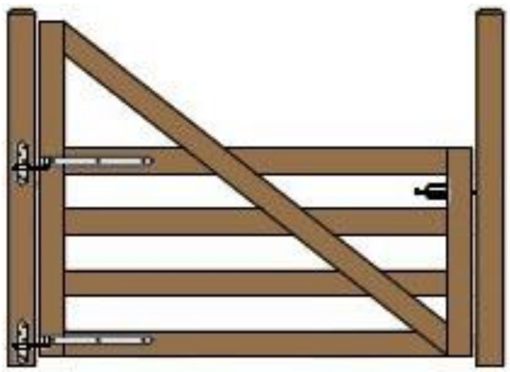
> ***Proof***
> Let F be on a line and G and G'' be on the line four units to each side of F. Let E be an intersection of circles of five-unit radii centered at G and G''. Observe that $\left|\overline{EF}\right| = 3$. By SSS, $\overline{EFG} \cong \overline{EFG''}$, which holds the equality $\angle EFG = \angle EFG''$. Thus, $\angle EFG = \rho$. ■

An analogous proof shows that a triangle of sides $5, 12$ and 13 is right; this was unknown to the Egyptians. Plumbers can use this triangle when installing 22.5° elbows. Integer solutions to the Pythagorean equation are known as Pythagorean triples. Students should be aware that Euclid devised a formula that generates Pythagorean triples: $u = m^2 - n^2,\ v = 2mn,\ w = m^2 + n^2$ for positive integers $m > n$. Verification is basic algebra; that ku, kv, kw for $k = 1, 2, \ldots$ gets them all is advanced. Try it with $n = 1$ and m even, or $n = 2$ and m odd.

$3 : 4 : 5$ right triangles are ubiquitous in *Common Core* because the programmers who compose their exams want to keep things neat by using only integers. *Varsity Tutors* Advanced Geometry Exam, problem #22 gives a rhombus of sides 5 units inscribed in a rectangle with height 4 units and asks the area. Problem 1.25 is the $3 : 4 : 5$ right triangle scaled up threefold and joined to the $5 : 12 : 13$ right triangle to be a $13 : 14 : 15$ triangle. A $13 : 20 : 21$ triangle has a 12-unit altitude for the same reason. A $15 : 20 : 25$ triangle is right – it is the $3 : 4 : 5$ right triangle scaled up fivefold; also, it is the threefold and fourfold $3 : 4 : 5$ right triangles joined along a 12-unit altitude; thus, it is the standard example of the geometric mean. Draw these triangles on your palm before exams and you have geometry mastered as *Common Core* defines the subject!

Basic Principles for Design of Wood and Steel Structures

As a geometer, you may be asked to design structures like gates, towers, gantries, or bridges.



Wrong!

Everybody knows that a diagonal is required to make a rigid triangle, but a drive through the country indicates that few know which way it goes. Wood beams can withstand a tremendous compressive load – 1700 psi for Douglas Fir – but cannot lift a load because the screws pull out. Steel is just the opposite; $\frac{1}{8}$" wire rope can lift 340 pounds, but stainless-steel tubes kink and fold under any large compressive load.[25]

Wooden diagonals go from the foot of the gate post upwards and wire rope diagonals go from the top of the gate post downwards. For a wooden tower to be rigid, it must have crossed wooden diagonals so there are some that are angled upwards towards any direction of wind.

A gantry is two A-frames with a beam between them and a hoist that slides along the beam. Mimicking the all-steel commercial ones with wood does not work because, when overloaded, the legs spread apart and bow outwards. The crossbar is pulling them together, which is not what wood does well. Make the base of the A-frame 81.65% of the leg length and attach wire ropes from the feet to the trisection points; the lower one will meet the leg at a right angle. Install a wooden crossbeam between the upper trisection points to tighten the wire rope against.

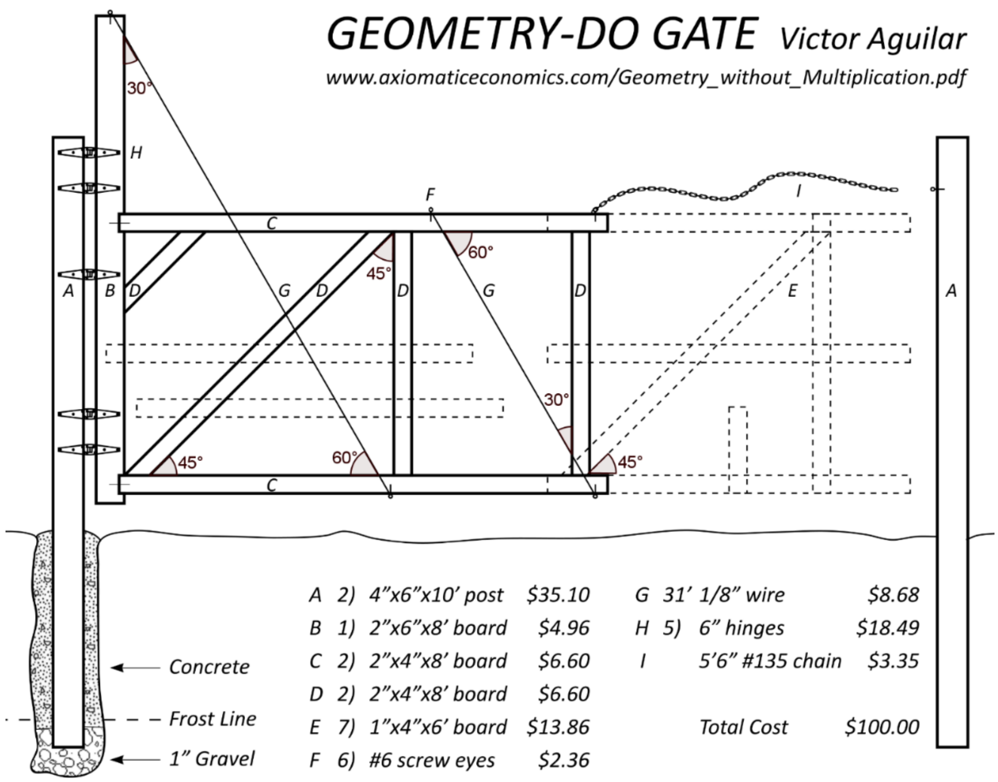
A drive through the country indicates that almost all wooden gates have collapsed. This is because they have a wooden diagonal angled downwards and it reaches across the entire $12'$ or $14'$ gate, making too horizontal an angle. Also, failed gates were over-engineered on the latch side, adding unnecessary weight far from the hinges. $1''$ planks are all it takes to stop cattle.

The gate shown below is $14'$ wide for farm equipment and is designed to stop cattle, not people. The wire rope loops through the eyes and around both sides of the gate. Solid lines are $2''$-thick boards or $4''$-thick posts, dashed lines are $1''$-thick planks. Note that the boards and posts are all assembled edgewise, so their widest sides are coplanar.

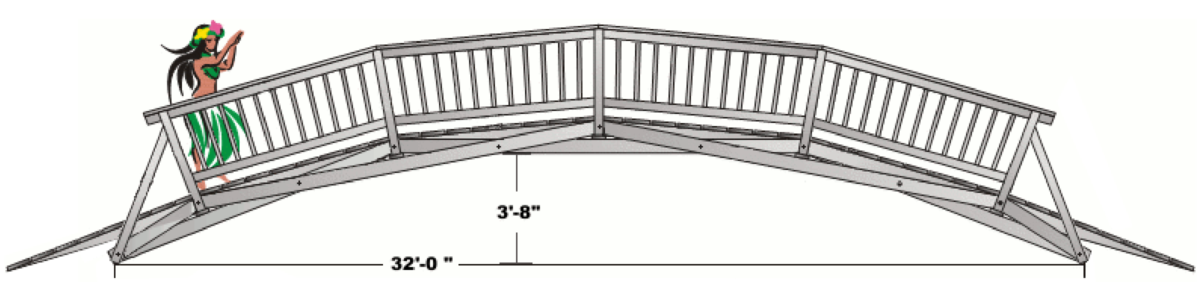
The C boards are inset into B and glued with wooden dowels to add strength. There are five hinges, and the gate post is rectangular; two hinges attached to a round post are weak. In the

[25] A single apostrophe means feet; a double apostrophe means inches.

winter, the ground freezes to the frost line and, in the spring, the top few inches thaw but do not drain through the frozen ground below, which is why it is so muddy. Water that soaks into the gate post can only drain out the bottom if it extends below the frost line. Also, there should be gravel, not concrete, below it to aid drainage. Gates often collapse because the post rots.



For automotive bridges too high to be supported with pillars, put a $4'' \times 6'' \times 12'$ post vertical and two $4'' \times 4'' \times 4'$ posts at $45°$ under the center of each of the two stringers and lift them with $0.5''$ steel cables attached to eye bolts in the concrete footers. The two vertical posts should have crossed braces – it is a mistake to look only at the side view and neglect twisting forces. Yellow belts will learn to build stone bridges cut from river rocks that can support truck traffic!

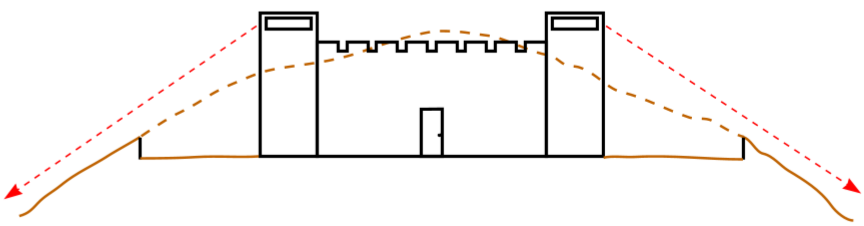


Detailed plans for wooden foot bridges of various sizes are available. Sorry Grasshopper, but, while the plans are free, the hula girl coming out to dance on your completed bridge is extra.

Defense Positioning and Geometry

Steel cannons with rifled bores brought an end to state-sponsored castle construction, but some of what was learned during the time of smoothbore bronze cannons is still relevant today for people engaged in low-intensity conflicts.[26] By low-intensity I mean, by mutual consent, both the home owners and the bandits restrict themselves to small arms, usually defined as 7.62 mm rifles and hand grenades, because they are under a real army, but it will ignore small-arms fire.

The **first principle** of building fortifications is to build them for yourself, not for the enemy. If you dig trenches or set out some Jersey barriers, they may stop the enemy's wheeled vehicles, but they will also provide cover for enemy infantry. Thus, you should have a vertical retaining wall facing rearward and an earthen glacis slope facing forward. Flatten the top of a gently rounded hill, digging deep enough that the cut-down area requires a three- to four-foot-high retaining wall all around it. The windows are high enough that the defenders can graze the slope with rifle fire, but the attackers cannot fire at the base of the house wall until they crest the retaining wall.



The **second principle** is to not have blind spots. The defenders should have bastions protruding from the corners of the building so attackers cannot press themselves up against the wall and be hidden from the windows. But, if the bastions are round, like the turrets in a medieval castle, there are blind spots directly in front of them. They should be tapered, like the points on a star.

The **third principle** is that stone shatters when hit by bullets, but concrete and brick do not. Also, landscape with crushed stone to make walking noisy. Get rid of boulders that can be thrown.

There is little application for geometry in the design of fortifications, but white-belt geometers should be familiar with the basics. Green belts will learn of machine gun emplacement, which really does require geometry. It would be embarrassing for the *Geometry–Do* practitioner to boast of these advanced techniques while showing ignorance of basics like glacis slopes.

[26] "Your home is like a fortress; no one comes in but the florist; the gardener and the maid." – *Pat Benatar*

Next we turn to the positioning of troops along a frontier that is plagued with cross-border raids.

Having heartily mocked the NES for getting the equation for a parabola wrong[27], let us be more positive and look to the work of someone who knows what the terms directrix, focus and latus rectum mean. Raj Gupta (1993) wrote a book about the most basic function of an army: defense against cross-border incursions. It often happens that the politicians and high-ranking officers have the wit to understand that an all-out war would be disasterous for both countries. But small units will cross the border to pillage; they have the tacit approval of their officers, but they also know that, if they get in trouble, their officers will not send anyone to rescue them.

It is reasonable to assume that both the bandits and the defenders move with equal speed over the same terrain. Thus, the set of points equally distant from the defenders' base and from the border are where the enemy can be met on the run; points inside this graph are where the defenders can reach first, giving them a few minutes to lay machine guns and find depressions in the dirt for riflemen to lie in; points outside this graph are where the enemy can get past the base and must be intercepted by soldiers from another base. Here we are using the perpendicular length theorem, which states that the perpendicular is unique and is the shortest segment from a point to a line. This is proven by yellow belts; so, if you are white belt, please read ahead now.

For simplicity, we will assume that the border is locally straight. In Cartesian coordinates, we will make it the x-axis and label the base's coordinates $(0, 2w)$ with $0 < w$.[28] Consider the parabola, $y - k = \frac{1}{4w}(x-h)^2$. The point midway between the base and the border is on the desired graph and it is w distant from each of them. Make this point the vertex of the parabola so $h = 0$ and $k = w$. The parabola is $y - w = \frac{x^2}{4w}$ or $y = \frac{x^2+4w^2}{4w}$. The vertex is equally distant from the base and the border. If we can prove this for every point on the parabola, then it defines the graph described in the previous paragraph. Thus, we must prove that the distance from (x, y) to $(0, 2w)$ is $y = \frac{x^2+4w^2}{4w}$, the distance from (x, y) to the x-axis. By the distance formula:

$$\sqrt{(x-0)^2+(y-2w)^2} = \sqrt{x^2+\left(\frac{x^2}{4w}-w\right)^2} = \sqrt{\frac{u^2}{16w^2}+\frac{u}{2}+w^2} \qquad \text{with} \quad u = x^2$$

$$= \frac{1}{4w}\sqrt{u^2+8w^2u+16w^4} = \frac{1}{4w}\sqrt{(u+4w^2)^2} = \frac{x^2+4w^2}{4w} = y$$

[27] See the preface, p. xv. For a more thorough kicking of the *National Council of Teachers of Mathematics*, read my review: www.researchgate.net/publication/335893456_Review_of_Essential_Understanding_of_Geometry

[28] Having the parabola upward and with $h = 0$ is the simplest case. $w < 0$ requires using absolute value; switching x and y turns the parabola sideways. The orange-belt appendix on linear algebra explains how to rotate and, in that case, you need to know that the distance from a point, (x_0, y_0), to a line, $Ax + By + C = 0$, is $\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$.

Proven! When the bandits were teenagers, they were too busy thieving to have learned how to factor a quadratic. They will surely regret sleeping through Algebra I when you get in front of them with five minutes to spare! That is *plenty* of time to lay a squad automatic weapon and for your riflemen to find depressions in the dirt where they can settle into their shooting positions.

The discussion above defines the parabola for a single army base located $2w$ klicks[29] from the border. But the more general question is, given the distance between army bases, how far should they be located from the border? The distance between the army bases is fixed by the budget; for instance, unless the politicians free up some more money, you may only have enough troops to man bases every four klicks. How far from the border should they be located to be sure that the bandits never get behind your line of bases? The parabolas of adjacent bases must intersect at or in front of the line of bases. So the question is, what must x be so that $y = 2w$? Since the parabolas of adjacent bases intersect halfway between them, the distance between bases is $2x$.

$$2w = \frac{x^2 + 4w^2}{4w} \quad \Rightarrow \quad 8w^2 = x^2 + 4w^2 \quad \Rightarrow \quad 4w^2 = x^2 \quad \Rightarrow \quad 2w = x$$

Thus, the distance between bases is $4w$, the width of each parabola when the bases are $2w$ from the border. In other words, the bases are twice as far from each other, $4w$, as they are from the border, $2w$. In the example above, if budget constraints require bases every four klicks, then the line of bases must be two klicks from the border. This is assuming that you know immediately when the border has been crossed – you are probably using electronic sensors – and your troops immediately move to intercept the bandits who are driving straight into your country. Of course, if Murphy has his way, none of these things will ever quite happen, so you will probably want to position your bases a little farther back – surrender a little more of your territory – to avoid allowing the bandits to ever get behind your line. Once into the interior, they are hard to catch.

Parabolas have other applications, so we must use more abstract language. The line that defines the border is called the **directrix**; the army base is at a point called the **focus**; and the segment parallel to the directrix that passes through the focus and has endpoints on the parabola is called the **latus rectum**.[30] For the parabola $y - k = \frac{1}{4w}(x - h)^2$, w is the distance from the vertex to either the focus or the directrix, $2w$ is the distance from the focus to the directrix, and $4w$ is the width of the latus rectum. For parabolas that were defined by the *National Evaluation Series* in which they got $\frac{1}{4w}$ upside down, the constant w is meaningless, as is most of what they teach.[31] Screeching, *"It's just a constant!!!"* is not a valid argument. We need to end *Common Core*.

[29] Klicks is an abbreviation for kilometers; clicks refer to increments of angle adjustment on an artillery piece.

[30] No giggling allowed when you hear the word rectum! It is Latin; it does not refer to any part of your anatomy.

[31] The NES and many textbooks use c, not w, but that is confusing because c is the constant term in parabolas.

The preceeding two pages are algebra and, if taken as a review of a past Algebra I course, it can be taught at any time. If the geometry teacher will be absent and must ask the algebra teacher to substitute, the algebra teacher can lecture from these two pages. Next comes a Euclidean construction; purists will teach it in orange belt, though many will teach all four pages together.

To make this really useful to a field officer, we must teach the compass-and-straightedge construction of a parabola. An army captain does not have a computer in his tent with graphic-design software that allows him to superimpose a parabola on a map image. And he does not have a color printer to print out this new map. All he actually has in his tent is a desk and a paper map. The rule about the bases being twice as far from each other as they are from the border is something he can remember. But to make this really useful, he must draw the parabolas on his map so his mortar gunners can treat the outside of the parabolas as a free-fire zone while his ground troops can be careful to stay inside their parabola. Also, they may unroll concertina wire along the parabolas to slow the enemy, but leave gaps where – after the mortar gunners have ceased fire – the ground troops can exit to pursue enemy troops back to the border.

This construction is Euclidean because it assumes that parallel lines are everywhere equidistant.

1. With a colored pen, mark the location of the base and, if the border is not perfectly straight, draw a straight line that follows the border as closely as possible.

2. Drop a perpendicular from the base to the border and locate its midpoint. This is the parabola vertex; mark it another color. Measure this length in centimeters and call it w.

3. With a pencil, draw a line parallel to the border through the parabola vertex. Then draw a series of lines parallel to this line each one centimeter apart and continue past the base.

4. Set your compass to $w + 1$ centimeters and, with pencil, draw arcs centered at the base that cut the first parallel line from the one that goes through the vertex. Mark these intersections with a dot of the same colored ink that was used for the vertex.

5. Repeat step #4 with a $w + 2$ cm arc intersecting the second parallel, then a $w + 3$ cm arc intersecting the third parallel, and so on for all the parallel lines.

6. Connect the dots with the same color of ink. You may want to free-hand this to make the parabola smoother than if it were composed of a series of straight segments.

7. Repeat this construction for each of the several bases in your area of operation.

The "twice as far from each other as they are from the border" rule depends on there being multiple units, like a four-platoon company. The captain decides where each platoon's base is to be built while the lieutenant in command of an individual platoon controls his troop's movements within their parabola and the shelling of targets outside their parabola. For instance, if the captain has four platoons and is tasked with guarding a straight length-M segment of the front line where $M = 16$ klicks, he will position bases 2 and 6 klicks inwards from the edges and 2 klicks back from the border. But what if he has only one unit to position in his area of operation?

Raj Gupta writes, "For an arbitrary probability density function of attack, all defending units must base themselves μ along the length-M front and distance σ inward from the border ($\sigma - \mu$ Theorem) in order to meet and defeat the invading forces as close to the front as possible (p. 8)." Proof of the $\sigma - \mu$ Theorem is beyond the scope of this book; but suffice it to say, μ and σ are the mean and standard deviation of the probability density function (pdf), respectively.[32] The pdf is an assignment of probabilities of the chance of attack at each point along the length-M front. The sum of the probabilities assigned to all the points on the front must add up to unity.

How are these probabilities assigned? This depends a lot on whether the enemy knows where your bases are. For the first line of defense, the platoon-size bases only two klicks from the border and featuring tall watch towers, it is obvious that they know. Thus, within each base's 4-klick wide subfront, there is zero probability of the enemy going hey diddle diddle, straight up the middle and a 50% probability of them attacking on either edge of the subfront. The mean is in the middle of the subfront, and the standard deviation is half the width of the subfront.

Thus, if each lieutenant were given the freedom of positioning his base anywhere behind his subfront regardless of what the other lieutenants are doing, logic would lead him and each of his fellow lieutenants to position their bases exactly as their captain would. Now suppose that the captain has a fifth platoon held in reserve. It does not have a watchtower and is positioned some distance from the front where it can move to reinforce any one of the four platoons.[33] Because the enemy does not know where it is, its probability density function is 20% at $0, 4, 8, 12$ and 16 klicks from one edge of its 16-klick front. Use your scientific calculator to find μ and σ, but do not use sample standard deviation like you would if you were doing confidence intervals. For this example, $\mu = 8$ klicks, the midpoint, and $\sigma \approx 5.657$ klicks back. Try this with different numbers of platoons. It is always optimal to have one reserve unit no matter how many bases there are; but the more bases, the closer the reserve can be to the front. If the enemy does not fear a platoon, they may invade anywhere; in this case, $\sigma = M/\sqrt{12} \approx 0.2887M$.

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[32] σ means a straight angle, but in statistics it is the symbol for standard deviation; on this page only, that is its use.

[33] Zulu chief Shaka had his reserves in a gully with their backs to the enemy and under orders that, if any turned to peer over the edge, they would be shot. He did this to prevent them from attacking before they were ordered to.

Yellow Belt Instruction: Congruence

Angle–Side–Angle (ASA) Theorem ***(Euclid, Book I, Prop. 26)***
Given two angles and the included side, a triangle is fully defined.

Proof
Given $\overline{EFG}$ and $\overline{JKL}$ with $\angle GEF = \angle LJK$, $\overline{EF} = \overline{JK}$ and $\angle EFG = \angle JKL$; let us assume that $\overline{GE}$ and $\overline{LJ}$ are not equal. Suppose that $\overline{GE} > \overline{LJ}$; there is a point M between G and E such that $\overline{ME} = \overline{LJ}$. $\overline{EFM} \cong \overline{JKL}$ by SAS. By congruence, $\angle EFM = \angle JKL$. But, by the interior angle axiom, $\angle EFM < \angle EFG$ because M is inside $\angle EFG$, which is given to be equal to $\angle JKL$; a contradiction. Suppose that $\overline{GE} < \overline{LJ}$. Proof that this is impossible is the same but has M between L and J. By trichotomy, $\overline{GE} = \overline{LJ}$. By SAS, $\overline{EFG} \cong \overline{JKL}$. ■

It is an easy corollary that isosceles triangles have two angle bisectors equal. Prove it, please.

Isosceles Triangle Theorem Converse ***(Euclid, Book I, Prop. 6)***
If two angles of a triangle are equal, then their opposite sides are equal.

Proof
Given $\overline{EFG}$ with $\angle E = \angle F$, $\overline{EFG} \cong \overline{FEG}$ by ASA and thus $\overline{GE} = \overline{GF}$. ■

Problem 2.1 *American engineers use a hybrid of English and metric lengths; inches divided into ten parts. Draw a segment $5.8''$, raise perpendiculars at each endpoint and bisect the right angles to form a triangle with the angle bisectors meeting at the apex. How long are the legs in 10th of an inch? Are you sure that they are equal? Are they the same length in hyperbolic geometry?*

Vertical Angles Theorem ***(Euclid, Book I, Prop. 15)***
Given $\overleftrightarrow{EF}$ and G, J on opposite sides of it, G, E, J are collinear iff a pair of vertical angles is equal.

Proof
Assume G, E, J are collinear in this order and F, E, K are collinear in this order. Then, $\angle GEF$ and $\angle GEK$ are supplementary. $\angle GEJ$ is straight; thus, $\angle GEF$ and $\angle JEF$ are supplementary. $\angle GEK$ and $\angle JEF$ are both supplementary to $\angle GEF$ and thus equal.

Assume F, E, K are collinear in this order; also, $\angle GEK = \angle JEF$. F, E, K are collinear and thus $\angle GEK$ and $\angle GEF$ are supplementary. By substitution, $\angle JEF$ and $\angle GEF$ are supplementary; hence, G, E, J are collinear. ■

Problem 2.2
Given $\overline{EFGH}$, if the diagonals bisect each other, prove that $\overline{EF} = \overline{GH}$ and $\overline{FG} = \overline{HE}$.

Exterior Angle Inequality Theorem ***(Euclid, Book I, Prop. 16)***
An exterior angle of a triangle is greater than either remote interior angle.

Given $\overline{EFG}$, extend $\overrightarrow{EF}$ past F to J. Prove that (1) $\angle EGF < \angle JFG$, and (2) $\angle GEF < \angle JFG$.

Proof

1. Extend $\overrightarrow{EM_{FG}}$ that much again to K, so M_{FG} is the midpoint of $\overline{EK}$. Connect F and K. By Pasch's axiom applied to $\overline{EFK}$ and $\overleftrightarrow{M_{FG}J}$, $\overleftrightarrow{M_{FG}J} \cap \overleftrightarrow{FK}$ is inside $\angle JFM_{FG}$. Hence, $\angle KFM_{FG} < \angle JFM_{FG}$ by the interior angle axiom. By the vertical angles theorem and SAS, $\overline{GM_{FG}E} \cong \overline{FM_{FG}K}$, and thus $\angle EGM_{FG} = \angle KFM_{FG}$. $\angle EGM_{FG} < \angle JFM_{FG}$ by substitution. Thus, $\angle EGF < \angle JFG$. •

Exterior Angle Inequality Theorem Corollaries ***(Euclid, Book I, Prop. 21)***

1. *The base angles of an isosceles triangle are acute.*
2. *A right or obtuse triangle has two acute angles.*
3. *Given $\overline{EFG}$ and P inside it, $\angle EGF < \angle EPF$.*

Greater Angle Theorem ***(Euclid, Book I, Prop. 18)***
If two sides of a triangle are unequal, then their opposite angles are unequal, the shorter side opposite the smaller angle and the longer side opposite the larger angle.

Proof

Given $\overline{FG} < \overline{EF}$ in triangle $\overline{EFG}$, find J between E and F such that $\overline{JF} = \overline{FG}$ and connect it to G. By the exterior angle inequality theorem, $\angle FEG < \angle FJG$. By the isosceles triangle theorem, $\angle FJG = \angle FGJ$. J is inside $\angle FGE$, so $\angle FGJ < \angle FGE$ by the interior angle axiom. Thus, $\angle FEG < \angle FJG = \angle FGJ < \angle FGE$. Simplifying, $\angle FEG < \angle FGE$. ■

Greater Side Theorem ***(Euclid, Book I, Prop. 19)***
If two angles of a triangle are unequal, then their opposite sides are unequal, the smaller angle opposite the shorter side and the larger angle opposite the longer side.

The word angle or side in the name refers to the result, not the given information.

Problem 2.3 *Diameters are the greatest chords. (They try not to let it go to their heads.) Proof?*

The triangle inequality theorem can go here. This is in an appendix, but we do have corollaries!

Triangle Inequality Theorem Corollaries

1. *Any side of a triangle is greater than the difference of the other two sides.*
2. *Given $\overline{EFG}$ and P inside it, $\overline{EP} + \overline{PF} < \overline{EG} + \overline{GF}$.*
3. *The sum of the medians is greater than the* ***semiperimeter*** *and less than the perimeter.*

Hinge Theorem ***(Euclid, Book I, Prop. 24, 25)***

If two triangles have two corresponding sides equal, the included angle in one is smaller/larger than in the other if and only if the opposite side is shorter/longer in the former than in the latter.

Orange belts study parallel lines, which are everywhere equidistant. For this statement to mean something, we must define the distance from a point to a line. There are many points on a line that define segments to a point not on the line. Which segment? The one of minimum length!

Define the **reflection** of G about $\overleftrightarrow{EF}$ as G_{EF} such that G, G', G_{EF} are collinear and $\overline{GG'} = \overline{G_{EF}G'}$. One can also reflect a line about a point; in problem 2.8 we reflect a highway about a farmhouse.

Perpendicular Length Theorem

The perpendicular is unique and is the shortest segment from a point to a line.

Part One

There is only one segment from a point to a line that is perpendicular to it.

Proof

Let J and G' be the feet of distinct perpendiculars dropped on $\overleftrightarrow{EF}$ from G with G' between J and F. By the right-angle postulate, $\angle GJF = \angle GG'F = \rho$. This contradicts the exterior angle inequality theorem, so the assumption that J and G' are distinct is not true. •

Part Two

The perpendicular is the shortest segment from a point to a line.

Proof

Let G' be the foot of the perpendicular dropped on $\overleftrightarrow{EF}$ from G, and J be another point on $\overleftrightarrow{EF}$. $\overline{GG_{EF}} < \overline{GJ} + \overline{JG_{EF}}$ by the triangle inequality theorem. By SAS, $\overline{GG'J} \cong \overline{G_{EF}G'J}$, so $\overline{GJ} = \overline{JG_{EF}}$. Halve both sides of $\overline{GG_{EF}} < \overline{GJ} + \overline{JG_{EF}}$ to get $\overline{GG'} < \overline{GJ}$. • ■

Perpendicular Length Theorem Corollaries

1. *Distinct perpendiculars raised from a line never intersect.*
2. *The hypotenuse is longer than either leg of a right triangle.*

Angle–Angle–Side (AAS) Theorem ***(Euclid, Book I, Prop. 26)***
Given two angles and a side opposite one of them, a triangle is fully defined.

Proof
Given $\overline{EFG}$ and $\overline{JKL}$ with $\angle E = \angle J$, $\angle F = \angle K$ and $\overline{FG} = \overline{KL}$; let us assume that $\overline{EF}$ and $\overline{JK}$ are not equal. Suppose that $\overline{JK} < \overline{EF}$; there is a point M between E and F such that $\overline{MF} = \overline{JK}$. $\overline{MFG} \cong \overline{JKL}$ by SAS; so, $\angle GMF = \angle LJK$. But $\angle LJK = \angle GEF$ is given, so $\angle GMF = \angle GEF$. $\angle GMF$ is an exterior angle of $\overline{EMG}$ equal to a remote interior angle of $\overline{EMG}$, which contradicts the exterior angle inequality theorem. Thus, $\overline{JK} < \overline{EF}$ is not true. Analogously, $\overline{EF} < \overline{JK}$ is not true. By trichotomy, $\overline{EF} = \overline{JK}$. By SAS, $\overline{EFG} \cong \overline{JKL}$. ■

Bill Gates[34] (*A Course Outline for Geometry*, p. 3) writes, "AAS is not sufficient for congruence." Should we tell him? He spent hundreds of millions of dollars to buy a monopoly. Will he listen?

Isosceles Altitudes Theorem
Two altitudes are equal if and only if the triangle is isosceles.

Proof
Given $\overline{EFG}$, assume that $\overline{EG} = \overline{FG}$. By AAS, $\overline{E'GE} \cong \overline{F'GF}$, so $\overline{E'E} = \overline{F'F}$.

Given $\overline{EFG}$, assume that $\overline{E'E} = \overline{F'F}$. By AAS, $\overline{GE'E} \cong \overline{GF'F}$, so $\overline{GE} = \overline{GF}$. ■

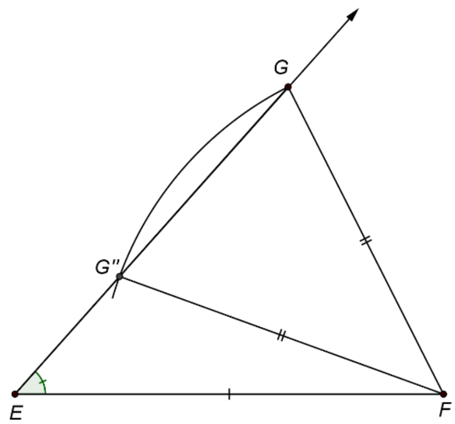
Hypotenuse–Leg (HL) Theorem
Given the hypotenuse and one leg of a right triangle, it is fully defined.

Proof
By C. 1.3, raise a perpendicular from a line the length of the leg and connect its endpoint to the line on both sides with segments the length of the hypotenuse. By the isosceles triangle theorem, the base angles are equal, and by AAS, the triangles are congruent. ■

Viviani Midpoint Theorem
A triangle is isosceles iff perpendiculars dropped from the base midpoint onto the sides are equal.

ASS cannot be a congruence theorem because two incongruent triangles, $\overline{EFG} \ncong \overline{EFG''}$, can be constructed with the same angle, side and side. But observe in the figure below that the counter-example requires $\angle E < \rho$. If $\angle E = \rho$, this is HL; if $\angle E > \rho$, it is OSS, which will be proven later.

[34] usprogram.gatesfoundation.org/-/media/dataimport/resources/pdf/2016/12/geometry-outline2014.pdf



Counterexample to the ASS Congruence "Theorem"

Faced with tremulous home-school parents who intended to teach geometry using his textbook but were unsure that they could, Givental[35] bolstered their courage by casting a magic spell:

> *Those reading these lines are hereby summoned to raise their children to a good command of Elementary Geometry, to be judged by the rigorous standards of the ancient Greek mathematicians.*

You will be pleased or perhaps appalled to learn that I too can cast magic spells… actually curses. The Aguilar Curse: Students who try to use the ASS "theorem" will grow donkey ears.

Johnny Geometer claims that any point P on $\overline{EF}$ bisects it! Let $\overline{EG} = \overline{FG}$. $\angle E = \angle F$ by the isosceles triangle theorem and $\overline{PG} = \overline{PG}$ by reflexivity. By ASS, $\overline{EGP} \cong \overline{FGP}$, so $\overline{EP} = \overline{FP}$. 😊

[35] In 2020 I said Kiselev, but I have since learned that the translator, Givental, is putting words in Kiselev's mouth.

This completes the basic geometry that even lame-brain *Common Core* students learn. The difference is that we prove these theorems and, hence, they must be presented in the correct order. *Common Core* students are just given a jumble of stuff to memorize in no particular order. Now that we have got the basics, let us use what we know to solve some practical problems!

Problem 2.4
Without a laser rangefinder, measure the distance across a river to construct a cable ferry.

> ***Solution***
> If E and P are posts on opposite banks, find F on $\overrightarrow{PE}$ inland of E. By P. 1.27, construct a rectangle $\overline{EFGH}$. Find P_1 on $\overrightarrow{HG}$ such that one can sight on P directly over a flag at M_{EH}. By ASA, $\overline{EM_{EH}P} \cong \overline{HM_{EH}P_1}$. So $\overline{EP} = \overline{HP_1}$, the length of cable needed. ■

This classic problem usually begins, "Construct $\overline{EH} \perp \overleftrightarrow{EP}$." But how, without the full rectangle? P. 2.4 is yellow belt because we could use a transit[36], but P. 1.27 is the inexpensive way to do it.

Problem 2.5
Use a transit to construct the corners of a house equidistant to a road concealed behind a fence.

> ***Solution***
> Remove a plank or drill a hole in the fence roughly perpendicular to the middle of the house. At the corner post, sight through the hole to the road. Put the transit there and measure the acute angle with the road. Move the transit to the corner post and lay off this angle. By AAS, the front of the house is on this ray. Fix the hole in the fence. ■

Lemma 2.1 ***(Euclid, Book I, Prop. 17)***
The sum of any two interior angles of a triangle is less than a straight angle.

> ***Proof***
> Consider $\angle E$ and $\angle F$ in $\overline{EFG}$ and let J be on $\overrightarrow{EF}$ past F. $\angle JFG$ is exterior to $\overline{EFG}$ at F. By the exterior angle inequality theorem, $\angle FEG < \angle JFG$. $\angle EFG + \angle JFG = \sigma$ by supplementarity and, by substitution, $\angle EFG + \angle FEG < \sigma$. ■

This lemma is absolute (neutral) geometry in the sense that Bolyai used the term to mean what is common to Euclidean and hyperbolic geometry. But it is not true in elliptic geometry. Some people use the term to include all three geometries. We will use the term in Bolyai's sense.[37]

---

[36] A transit is an optical device mounted on a tripod that measures angles. A theodolite is too, but more expensive.
[37] Bolyai said "absolute" in 1832, but geometers prefer "neutral" today; neither has ever meant elliptic geometry.

Angle–Side–Longer Side (ASL) Theorem
Given an angle and the side opposite the angle not less than a near side, a triangle is fully defined.

Proof
Given $\overline{EFG}$ with $\overline{EF} \leq \overline{FG}$ and $\overline{JKL}$ with $\angle J = \angle E$, $\overline{JK} = \overline{EF}$ and $\overline{KL} = \overline{FG}$, assume that $\overline{JL} \neq \overline{EG}$; specifically, $\overline{JL} < \overline{EG}$. Thus, there is a M between E and G such that $\overline{EM} = \overline{JL}$. By SAS, $\overline{EFM} \cong \overline{JKL}$, which holds the equality $\overline{FM} = \overline{KL}$. But $\overline{KL} = \overline{FG}$ is given; thus, $\overline{FM} = \overline{FG}$ by transitivity and $\angle FMG = \angle FGM$ by the isosceles triangle theorem. By lemma 2.1, their sum is less than a straight angle and, since they are equal, they are acute; hence, $\angle EMF$ is obtuse by supplementarity. In $\overline{EFM}$, the obtuse angle $\angle EMF$ is larger than both $\angle MEF$ and $\angle MFE$. By the greater side theorem, $\overline{EF}$ is the longest side; that is, $\overline{FM} < \overline{EF}$, which contradicts the assumption that $\overline{EF} < \overline{FG} = \overline{FM}$. Thus, $\overline{JL} \neq \overline{EG}$ is wrong; by SSS, $\overline{EFG} \cong \overline{JKL}$. ■

Given $\overline{EFG}$ and $\overline{JKL}$ with $\angle E = \angle J$, $\overline{EF} = \overline{JK}$ and $\overline{FG} = \overline{KL}$, if $\angle G$ and $\angle L$ are both obtuse, right or acute, then $\overline{EFG} \cong \overline{JKL}$. This is an alternative way to state the ASL theorem. Benjamin Catalfo used ASL to prove that a problem on the *Regents Common Core* geometry exam was wrong.[38]

Obtuse Angle–Side–Side (OSS) Theorem
Given an obtuse angle and two sides that are not bracketing it, an obtuse triangle is fully defined.

Proof citing ASL
By lemma 2.1, the obtuse angle is larger than either of the other angles so, by the greater side theorem, the side opposite it is longer than either other side. Thus, by ASL. ■

Proof citing HL
By supplementarity, the angle exterior to the given angle is fully defined. Drop a perpendicular from the vertex between the given sides to the extension of the unknown side. It intersects the extension because the given angle is obtuse and forms a triangle outside the given triangle that, by AAS, fully defines the length of the altitude. By HL, the union of the two triangles is fully defined and, by subtraction, the given one is. ■

Angle Bisector Theorem
A point is on an angle bisector if and only if it is equidistant from the sides of the angle.

Proof
Given the angle bisection, the point is equidistant by AAS; the converse by HL. ■

[38] www.newsday.com/long-island/education/li-educators-criticize-revamped-regents-geometry-exam-1.13803229

Mid–Term Exam

In *Geometry–Do*, the final exam counts for 100% of students' grades. So, this mid-term does not count, but it does hint at the nightmarish exam lying in wait for them at the end of the semester!

Problem #1

Given $\overline{EFG}$ such that $\overline{EG} < \overline{FG}$, recall that G' is the foot of the altitude from G onto $\overleftrightarrow{EF}$, G^* is the intersection of the $\angle G$ bisector with $\overline{EF}$, and M_{EF} is the midpoint of $\overline{EF}$. Prove that:

1. $\angle GG^*E < \angle GG^*F$
2. $\overline{G^*E} < \overline{G^*F}$
3. *G^* is between G' and M_{EF}*

Problem #2

Given $\overline{PQG}$, construct $\overline{EFG}$ isosceles so the apex angles $\angle PGQ \equiv \angle EGF$ and $M_{PQ} \in \overline{EF}$. Prove that $\overline{EP} = \overline{FQ}$.

Problem #3

Suppose problem #2 had read, "Given $\overline{EFG}$ isosceles and M on its base, $\overline{EF}$, find P on $\overrightarrow{GE}$ and Q on $\overrightarrow{GF}$ such that M is the midpoint of $\overline{PQ}$." Can you do this construction without assuming the conclusion you will be asked to prove in part two, $\overline{EP} = \overline{FQ}$?

Problem #4

$\overline{EFG}$ is isosceles with base $\overline{EF}$. Let $P \neq M_{EG}$ be on $\overline{EG}$ and Q be on $\overline{FG}$ such that $\overline{GP} + \overline{GQ} = \overline{GE}$. Prove that the mid-segment, $\overline{M_{EG}M_{FG}}$, bisects $\overline{PQ}$.

Here are some pop quiz problems that can be inserted earlier in yellow belt. The weak students who are slayed by a pop quiz are unlikely to be reading ahead, so they will probably be surprised.

Pop quiz about the **exterior angle inequality theorem!** *Prove that $\varphi < \rho$.*

Pop quiz about the **triangle inequality theorem** and its corollaries!
Prove that, if the bisector of $\angle G$ bisects the perimeter of $\overline{EFG}$, then $\overline{EFG}$ is isosceles.

Pop quiz about the **perpendicular length theorem**, its corollaries, and **AAS!**

1. *Given P not on $\overleftrightarrow{EF}$ and P' its perpendicular foot on $\overleftrightarrow{EF}$, if $\overline{P'E} < \overline{P'F}$, then is $\overline{PE} < \overline{PF}$?*
2. *Prove that the sum of the legs of a right triangle is less than twice the hypotenuse.*
3. *Prove that, if two triangles are congruent, then their corresponding altitudes are equal.*

Intermission (Johnny Geometer's Big Invention)

Every geometry teacher everywhere has announced that trisecting an angle is impossible with compass and straightedge, at which point a student's hand popped up in the air and he suggested, "Make it the apex of an isosceles triangle and trisect the base." The teacher then drew an obtuse isosceles triangle with its base trisected; visual inspection made it clear that the outer angles equal each other, but they are much smaller than the center angle.

But this left the student with too much hope.[39] Unheard by the teacher, the student walked out of class muttering to himself, "She didn't even give my idea a chance! Maybe it works for acute angles?" Then he went hungry because he ignored his food and spent his lunch hour squinting at a series of isosceles triangles with increasingly narrow apex angles. Lacking a protractor (they are not allowed in intrinsic geometry), he struggled to see that his three angles were not all equal.

Problem 2.6 *Johnny Geometer claims that an arbitrary angle can be trisected by making it the apex of an isosceles triangle and then trisecting the base! Can you prove him wrong?*

Solution

Given $\overline{EFG}$ isosceles with the apex $\angle G$, by C. 3.9 or C. 3.11 trisect the base with trisection points J near E and K near F. Connect $\overline{GJ}$ and $\overline{GK}$. Draw a circle centered at G that passes through J and K. By the continuity theorem, it cannot also pass through E, so there is a point L on $\overline{EG}$ such that $\overline{GL} = \overline{GJ} = \overline{GK}$. By exterior angle inequality theorem corollary #1, $\angle GLJ$ is acute, so its supplement $\angle ELJ$ is obtuse and thus $\angle LEJ < \angle ELJ$. By the greater side theorem $\overline{LJ} < \overline{EJ} = \overline{JK}$. By the hinge theorem, $\angle LGJ < \angle JGK$. ■

There is an angle trisection technique that requires putting two scratches on your straightedge an arbitrary distance apart. Most geometry textbooks will tell you that scratching your straightedge is not allowed in intrinsic geometry, but this is not true. You can scratch it all you want, including using your compass to lay off a series of equidistant scratches to form a crude ruler.[40] White belts did this to prove the Egyptian triangle theorem.[41] What we are not allowed to do is assign real numbers to lengths as Birkhoff did with his ruler postulate. Real numbers –

[39] We are not trying to crush hope – though that is fun – we just want their hopes directed towards plausible goals.

[40] Moise & Downs (p. 530) write, "The angle-trisection problem becomes solvable if we relax the rules very slightly by allowing ourselves to make two marks on the straightedge." This makes geometry sound like a parlor game with arbitrary rules designed to confound players. In the 2300 years since Euclid, for the first 2250 years rulers did not exist because it requires a CNC machine to scratch steel in 0.1 cm increments. They were doing the best that they could. The carpenter's square shown on the next page did not exist before World War Two.

[41] The ancient Egyptians had aqueducts, but they did not use $22.5°$ elbows because they had only six scratches on their straightedges, not the 14 needed to construct $5 : 12 : 13$ right triangles, as I instruct modern plumbers to do.

are infinitely close to each other and no machinist can or ever will put infinitely close scratches on a straightedge, so there is no physical reality to Birkhoff's ruler; it is a fictional devise. Yellow belts can skip the orange-belt proof; being Euclidean is not what is wrong with this construction.

Construction 2.1 *Trisect an angle.* ***(This is not a real geometry construction!)***

Solution

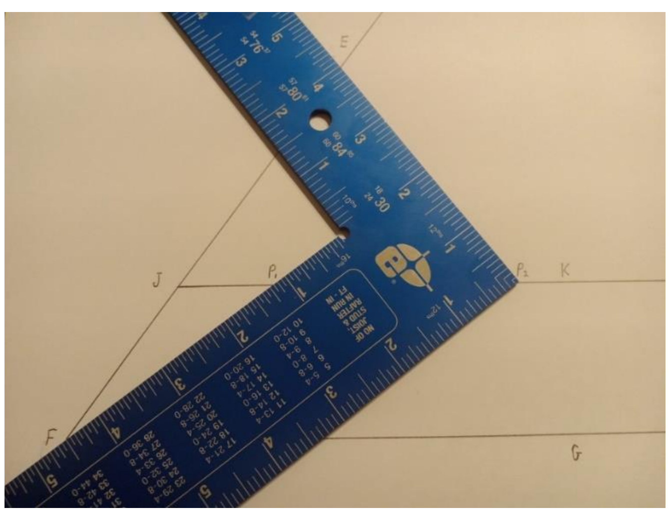
Given $\angle EFG$ arbitrary and a straightedge with scratch marks r apart, draw a circle, ω, of radius r centered at F. Let $E'' \coloneqq \overrightarrow{FE} \cap \omega$ and $G'' \coloneqq \overrightarrow{FG} \cap \omega$. Extend $\overrightarrow{GF}$ past ω to a point P such that $Q \coloneqq \overline{PE''} \cap \omega$ and $\overline{PQ} = r$. Then $\angle QPF = \frac{1}{3}\angle EFG$ and, by C. 1.5, it can be twice replicated inside $\angle EFG$ to trisect it. ■

Proof ***(The exterior angle theorem is beginner orange belt.)***

Apply the exterior angle theorem to $\overline{QFP}$, which is isosceles, so $\angle E''QF = 2\angle QPF$. Apply the isosceles triangle theorem to $\overline{E''QF}$, and the exterior angle theorem to $\overline{FE''P}$. ■

So, if scratching the straightedge is not what is wrong with this construction, then what is? Finding P requires nudging the scratched straightedge around until it makes $Q \coloneqq \overline{PE''} \cap \omega$ and $\overline{PQ} = r$ true. This is trial and error! Trial-and-error is what computer programmers do – they call it a linear search – and nobody ever mistook a computer programmer for a geometer.

There is another angle trisection technique employed by cabinet makers when they must trisect an angle. Try it! The need to nudge your square here and there until it fits makes it clear that this is trial and error. I leave it as an (orange belt) exercise for the student to prove that it works.



And now, circles! I love circles! So round and plump! Nothing like triangles with their bony hips!

Chord Inside Circle Theorem ***(Euclid, Book III, Prop. 2)***
Given a circle and any two points on it, the ***chord*** *between the points is entirely inside the circle.*

Proof
Given $\overline{EF}$, a chord of a circle centered at O, let M be inside $\overline{EF}$. $\angle OEM = \angle OFM$ by the isosceles triangle theorem. $\angle OMF > \angle OEM$ by the exterior angle inequality theorem, so $\angle OMF > \angle OFM$. By the greater side theorem, $\overline{OF} > \overline{OM}$. Thus, the result. ■

Diameter and Chord Theorem ***(Euclid, Book III, Prop. 3)***
A ***diameter*** *bisects a chord if and only if the diameter is perpendicular to the chord.*

Proof
Given chord $\overline{EF}$ in a circle with center O, $\overline{OM_{EF}}$ is on a diameter. $\overline{OM_{EF}E} \cong \overline{OM_{EF}F}$ by SSS. By the mediator theorem, the diameter is perpendicular to the chord.

If $\overleftrightarrow{OM} \perp \overleftrightarrow{EF}$ with M on chord $\overline{EF}$, then, by HL, $\overline{OME} \cong \overline{OMF}$. Thus, $\overline{ME} = \overline{MF}$. ■

Diameter and Chord Theorem Corollaries ***(Euclid, Book III, Prop. 9, 10)***
1. *Given a circle with center O and E, F, T on the circle such that $\overleftrightarrow{EF} \perp \overleftrightarrow{OT}$, then $\overline{ET} = \overline{FT}$.*
2. *If more than two equal segments can be drawn to a circle from a point, it is its center.*
3. *If two circles intersect more than twice, then they coincide and so intersect everywhere.*
4. *If every possible mediator of segments with endpoints chosen from among three or more points are concurrent, then these points are all concyclic.*

Equal Chords Theorem ***(Euclid, Book III, Prop. 14)***
In the same or equal circles, equal chords are equally distant from the center, and the converse.

Proof
Let $\overline{EE''} = \overline{FF''}$ be chords in a circle with center O; let E', F' be the feet of perpendiculars dropped on them from O, respectively. By the diameter and chord theorem, $\overline{EE'} = \overline{FF'}$. By HL, $\overline{OE'E} \cong \overline{OF'F}$, which holds the equality $\overline{OE'} = \overline{OF'}$. •

Proof of the converse is left as an exercise.

Unequal Chords Theorem ***(Euclid, Book III, Prop. 15)***
Of two chords in a circle, the one nearer the center is longer; and the longer is nearer the center.

Shortest Chord Theorem
The shortest chord through a point in a circle is perpendicular to the diameter through that point.

Lemma 2.2
A line intersects a circle in at most two points.

Proof
Let E, F, G be intersection points of a line and a circle with F inside $\overline{EG}$. E and G are on the circle, so $\overline{EG}$ is a chord. By the chord inside circle theorem, F is inside the circle. ■

Tangent Theorem ***(Euclid, Book III, Prop. 18, 19)***
A line intersects a circle where it is perpendicular to the radius iff that is a ***touching point****.*

Proof
Let F be a point of intersection perpendicular to the radius and O be the circle center. Suppose the line and the circle also intersect at E. Find G on the line such that $\overline{EF} = \overline{FG}$. By SAS, $\overline{EFO} \cong \overline{GFO}$, so $\overline{EO} = \overline{GO}$; G is on the circle. Thus, E, F, G are distinct points on a line and on a circle, a contradiction by lemma 2.2, so F is a touching point. •

Assume that the line intersects only once; every other point on the line is outside the circle and thus farther from the center than the radius. By the perpendicular length theorem, only the perpendicular is the shortest segment from a point to a line. • ■

Common Chord Theorem
If two circles have a ***common chord****, its mediator is the* ***line of centers****.*

Proof
By the diameter and chord theorem, the mediator of the chord is a diameter to both circles and on the line of centers. ■

Construction 2.2 ***(Euclid, Book III, Prop. 17)***
Through a point outside a circle, draw a line ***tangent*** *to the circle.*

Yellow Belt Solution
Let O be the center of the given circle, P be the given outside point and M be the intersection of $\overline{OP}$ with the circle. Draw a **concentric** circle that passes through P. By C. 1.3, raise a perpendicular from M and call an intersection with the larger circle Q. Let N be the intersection of $\overline{OQ}$ and the given circle; N is the desired touching point. ■

Proof
By SAS, $\overline{MOQ} \cong \overline{NOP}$, which holds the equality $\angle M = \angle N$. $\angle M$ is right by construction, so $\angle N$ is right and, by the tangent theorem, N is a touching point. ■

When the point is far from the circle (*e.g.,* a machine gun and its kill circle), the outer circle may go over the edge of the paper. Also, one's compass may not be capable of drawing this big circle. The green-belt solution, C. 4.4, requires an arc only half this radius and extending no farther than the given circle, so it will be the standard technique. But, even if yellow belts do not have the most efficient method, they have a method, so everything we describe in this chapter is feasible.

Common Point Theorem ***(Euclid, Book III, Prop. 11, 12)***
An intersection of two circles is a touching point if and only if it is on the line of centers.

Proof
If it is a touching point, then, by the tangent theorem, it is perpendicular to the radii from both centers and, by supplementarity, it is on the line of centers. If it is on the line of centers, then, by the perpendicular length theorem, the tangent there is unique. ■

Two Tangents Theorem
Two tangents from an external point are equal and their angle bisector intersects the center.

Proof
By the tangent theorem, the touching points are where the radii are perpendicular to the tangents and so, by the perpendicular length theorem, the center is equidistant from the tangents. By the angle bisector theorem, their angle bisector intersects the center. By HL, the two triangles are congruent. Hence, the tangents are equal. ■

Tangent Bisection Theorem I
If two circles touch, the perpendicular to the line of centers through the circles' touching point cuts their common tangents in half.

Blue belts will prove tangent bisection theorem II about the extension of the common chord.

Mirror Problem
Find the point on a mirror to shine a laser at a target.

Solution
Let J be the laser and K be the target. Drop perpendiculars from J and K to the mirror or its extension with feet J' and K'. Extend $\overrightarrow{KK'}$ an equal distance to $K_{J'K'}$ the reflection of K about $\overleftrightarrow{J'K'}$. Let $P := \overline{JK_{J'K'}} \cap \overleftrightarrow{J'K'}$. By SAS, $\overline{KK'P} \cong \overline{K_{J'K'}K'P}$, so $\angle KPK' = \angle K_{J'K'}PK'$. By the vertical angles theorem, $\angle K_{J'K'}PK' = \angle JPJ'$. By transitivity, $\angle KPK' = \angle JPJ'$, so aim for P; the angles of incidence, $\rho - \angle JPJ'$, and of reflection, $\rho - \angle KPK'$, are equal. ■

Note the assignment operator and the intersection symbol in $P := \overline{JK_{J'K'}} \cap \overleftrightarrow{J'K'}$. Lasers have the angles of incidence and of reflection equal, but bullets do not; they skim just over the asphalt.

A reflection is a point defined by another point and a line; the target is the reflection of the point behind the mirror that the laser is aimed at, and vice-versa. Because the two points are the same distance from the mirror, reflection is an **isometry**; a **transformation** that preserves distance. M_{EF}, M_{FG}, M_{EG} are midpoints; in a **tangential quadrilateral**, $I_{EF}, I_{FG}, I_{GH}, I_{HE}$ are incircle touching points. These and Thébault's notation are exceptions to double subscripts denoting reflections.

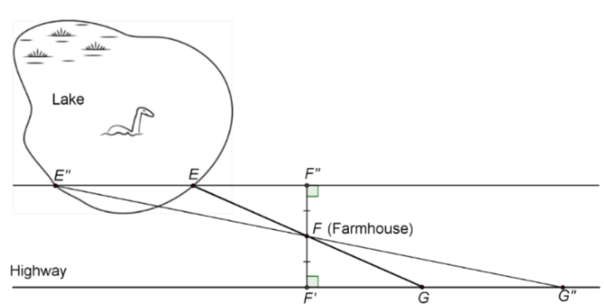
Problem 2.7 *Two towns are on the same side of a straight railroad track and some distance away. Where should a railway station be built to minimize the sum of the roads to the two towns?*

> ***Solution***
> Guess at where the station should be and draw in the roads. Draw a road from the station to one town's reflection. By SAS the reflected triangles are congruent and so their hypotenuses are equal. By definition of segment, if the station is not collinear with one town and the other town's reflection, it is badly guessed. Correct it. ■

The roads are a physical representation of the laser beam. Thus, the **mirror property** is that reflections preserve distance, so the laser beam in the mirror problem travels the same distance if it bounces off the mirror to the target or if it penetrates and goes to the target's reflection. To reflect a line across a point, reflect two points on the line across it and draw a line through them. One can choose any two points; but, if we take one to be the foot of the perpendicular, we get another method for constructing the reflection, which is sometimes taken as its definition.

Line Reflection Theorem
Two lines are reflections across a point iff the perpendicular dropped from that point onto one line, if extended in the opposite direction an equal distance, meets the other line at a right angle.

Problem 2.8 *There is a roughly circular lake, a straight highway, and an abandoned farm. You have purchased the farm with the idea of turning the farmhouse into a way station for fishermen. Pave a straight road to the lake so the farm is at its exact midpoint. Discuss the possibility of this.*



Problem 2.9 *Given two circles on opposite sides of a line, construct an equilateral triangle with one vertex on the line and the other two vertices on each of the two circles.*

Johnny Geometer insists that there are an infinity of solutions to P. 2.9, not just 0, 2 or 4. Yes?

Problem 2.10 *Given $\angle EFG$ acute and P within it, find points on each ray such that the perimeter of the triangle they make with P is minimal.*

Solution

Let P_{FE} and P_{FG} be reflections of P about $\overrightarrow{FE}$ and $\overrightarrow{FG}$, respectively. $J := \overline{P_{FE}P_{FG}} \cap \overrightarrow{FE}$ and $K := \overline{P_{FE}P_{FG}} \cap \overrightarrow{FG}$. $\overline{PJK}$ has perimeter $\overline{P_{FE}P_{FG}}$, which is of minimal length. ■

Constructing a triangle so its perimeter is minimal will become the bread and butter of the *Geometry–Do* practitioner; he should be able to just look at a problem like P. 2.10 and immediately construct the solution. But the thought process of guessing at the solution and then seeing what is wrong with the picture so it can be redrawn correctly should not be forgotten. (Here, if we had guessed at J and K, then drawn $\overleftrightarrow{JK}$, laid off $\overline{PJ}$ past J to P_{FE} and laid off $\overline{PK}$ past K to P_{FG}, our "solution" would violate the perpendicular length theorem.)

If the student becomes too blasé about the initially easy problems, he will later encounter a difficult problem and feel like he has run into a wall. But, if he thinks carefully on the easy problems and treats each one as training for more difficult problems that he knows are coming, he will later discover himself doing problems that other geometers consider difficult without having ever noticed his own passage.

Minimal Base Theorem

Given the apex angle and the sum of the legs, the triangle with minimal base is isosceles.

Proof

Construct $\overline{EFG}$ so $\angle G$ is the given apex angle and $\overline{GE} = \overline{GF}$ are each half the given sum of legs. Position J and K so E is between G and J, K is between G and F, and $\overline{EJ} = \overline{FK}$. $\overline{GE} + \overline{GF} = \overline{GJ} + \overline{GK}$. Drop perpendiculars from J and K to $\overleftrightarrow{EF}$ with feet J' and K', respectively. $\angle J'EJ = \angle K'FK$ by the isosceles triangle theorem and the vertical angles theorem. By AAS, $\overline{J'EJ} \cong \overline{K'FK}$, which holds the equality $\overline{EF} = \overline{J'K'}$. By perpendicular length theorem corollary #2, $\overline{J'K'} < \overline{JK}$; thus, $\overline{EF}$ is the shortest possible base. ■

Common Core textbooks are fond of the acronym CPCTC, which stands for Corresponding Parts of Congruent Triangles are Congruent. I do not use this acronym for two reasons:

1. It is wrong. It should be Corresponding Magnitudes of Congruent Triangles are Equal, or CMCTE. "Parts" is nowhere defined; they are magnitudes. Using the term congruent for both triangles and magnitudes to the complete exclusion of the term equal is wrong. Congruence and equality are different things; there is good cause for having two words.

2. It is excessively verbose. When I say, "by SAS, $\overline{EFG} \cong \overline{JKL}$ and so $\overline{GE} = \overline{LJ}$," the students know immediately from how the vertices are ordered that $\overline{EF} = \overline{JK}$ and $\angle EFG = \angle JKL$ and $\overline{FG} = \overline{KL}$. Sometimes I will say, "$\overline{EFG} \cong \overline{JKL}$, which holds the equality $\overline{GE} = \overline{LJ}$." Students know that we say that two triangles are congruent because congruence implies that corresponding magnitudes are equal; they do not need to be repetitively told this.

Problem 2.11 *Through one of the two points of intersection of two equal circles, draw two equal chords, one in each circle, forming a given angle.*

Solution
Bisect the given angle and replicate this on both sides of the common chord. The sides of this angle extended make two chords, one in each circle, forming the given angle. ■

Proof
Let O_1, O_2 be the centers, $\overline{FG}$ their common chord and F_1, F_2 the endpoints of the constructed chords $\overline{FF_1}$, $\overline{FF_2}$ in circles around O_1, O_2, respectively. By the common chord theorem and HL, $\overline{O_1FM_{FG}} \cong \overline{O_2FM_{FG}}$, and so $\angle O_1FM_{FG} = \angle O_2FM_{FG}$. By angle addition, $\angle F_1FG = \angle O_1FM_{FG} \pm \angle O_1FF_1$ and $\angle F_2FG = \angle O_2FM_{FG} \pm \angle O_2FF_2$. By angle addition, $\angle F_1FO_1 = \angle F_2FO_2$. By the diameter and chord theorem, $\angle O_1M_{F_1F}F$ and $\angle O_2M_{F_2F}F$ are right. By AAS, $\overline{M_{F_1F}FO_1} \cong \overline{M_{F_2F}FO_2}$, so $\overline{M_{F_1F}F} = \overline{M_{F_2F}F}$. Doubling, $\overline{FF_1} = \overline{FF_2}$. ■

Wasn't this proof made a lot easier by saying "and so" than by citing CMCTE? In *How Math Can Be Taught Better*[42], I mock a *Common Core* textbook that takes five steps to prove that two angles, both given as right, are equal. *Duh! Common Core* textbook authors purposefully make proofs as boring and tedious as possible, so nobody will lament their loss in the remaining 95% of the textbook when the author just starts announcing theorems without proving them. *Clever!*

Problem 2.12 *Through one of the two points of intersection of two circles, draw a line that makes equal chords in the two circles.*

Problem 2.13 *Through three concentric circles, draw a line that they cut into two equal segments.*

[42] www.researchgate.net/publication/282947903_How_Math_Can_Be_Taught_Better

Let us now illustrate another method for solving geometry problems: Solve the problem with algebra and then replicate the algebra with geometry.

Construction 2.3 *Construct two segments given their sum and their difference.*

> ***Solution***
> Add the sum and the difference. Bisect this to get the longer segment and then cut off the difference to get the shorter segment. ■

$(x+y)+(x-y)=2x$. Thus, we add the sum and the difference and then bisect it to get the longer segment. $x-(x-y)=y$. Thus, we cut off the difference to get the shorter segment.

For the most part I despise *Common Core*, but one practice that I will adopt is their insistence that math classes include some classic literature. Who has not read *The Pit and the Pendulum?*

> *I could no longer doubt the doom prepared for me by monkish ingenuity in torture… [The pendulum's] nether extremity was formed of a crescent of glittering steel, about a foot in length from horn to horn; the horns upward, and the under edge evidently as keen as that of a razor.* – Edgar Allen Poe

Problem 2.14 *If the horns of Poe's pendulum are at points E and F one moment and then at points E'' and F'' a minute later, where is the axle from which the pendulum is suspended?*

> ***Solution***
> By the diameter and chord theorem, the mediators to chords $\overline{EE''}$ and $\overline{FF''}$ are diameters. Their intersection is the common center of the two concentric circles. ■

Note that $\overline{EF}$ and $\overline{E''F''}$ are not chords; there is no assurance that the horns rise equally above the blade. The endpoints of each chord are the same vertex of the figure at different times. Each point on the pendulum makes its own circle around an axle common to them all.

We next introduce our first triangle center! Other centers exist for acute triangles and, while we cannot define this precisely, for almost acute triangles, but they require the parallel postulate for general triangles. The medial point exists in neutral geometry, but it is deferred to orange belt.

Three highways cross farmland and intersect to form a triangle. People have complained that cell phone service is poor in this area, so the cell phone company intends to build a tower equidistant from the three highways. Where should they build it to best serve their customers?

Incenter Theorem ***(Euclid, Book IV, Prop. 4)***
The bisectors of a triangle's interior angles are concurrent at an interior point, the ***incenter****, I.*

Proof
Given $\overline{EFG}$, by the crossbar theorem, E^* is inside $\overline{FG}$; thus, $\overline{EE^*}$ is inside $\overline{EFG}$. By the crossbar theorem applied to $\overline{EFE^*}$, the bisector of $\angle F$ cuts $\overline{EE^*}$ and this intersection, I, is interior to $\overline{EFG}$. By the angle bisector theorem applied to $\angle E$ and to $\angle F$, respectively, I is equidistant from $\overline{EG}$ and $\overline{EF}$ and from $\overline{FE}$ and $\overline{FG}$; thus, by transitivity, I is equidistant from $\overline{EG}$ and $\overline{FG}$. By the angle bisector theorem, I is on the bisector of $\angle G$. ■

The **incircle** is around the incenter and touches the sides at I_E, I_F, I_G. The verb "inscribe" refers to drawing a circle in a triangle, never a triangle in a circle; this construction is not fully defined.

A city park is in the corner of two roads that intersect at an arbitrary angle; in this park, the VFW has erected a statue of a soldier and the city government has erected a statue of the town's founder. They wish to split the cost of a flagpole and position it where it is equally visible to drivers on both roads, and it is also equidistant from their respective monuments. Where?

Problem 2.15
Given two points inside an angle, find a point equidistant from the points and from the rays.

Military cadets should know what the words enfilade and defilade mean. To enfilade a road or a trench is to fire down the length of it. This is important to machine gunners because it is difficult to **traverse** a gun to lead a moving vehicle; guns only work well if the enemy is coming directly at you or – even better – directly away from you. Also, this is the only way to get bullets inside a trench; firing across it is pointless. Grenadiers can traverse their launch tubes more easily than machine gunners can traverse, so they fire from defilade; that is, from a concealed position alongside a road. But grenades are slow, so they must get close enough to lead moving vehicles.

The following theorem shows that an equilateral triangle with an important point at the center (*e.g.,* a munitions dump) is the best defense against both enemy aircraft and enemy troops if you have only three anti-aircraft guns. This motivates the Dakota defense problem, to be solved later.

In neutral geometry, the **circumcenter** exists for acute triangles, which is sufficient for the following theorem. The verb "circumscribe" always refers to drawing a circle around a triangle.

Incenter and Circumcenter Theorem
A triangle is equilateral if and only if its incenter and its circumcenter coincide.

Laying ambushes is great fun (unless, of course, you get killed), but we cannot immediately put heavy machine guns in the students' hands. We have more geometry to learn, so let's get busy!

Incircle Theorem
Given $\overline{EFG}$, then twice $\overline{I_G M_{EF}}$ is the absolute difference of $\overline{FG}$ and $\overline{GE}$.

Proof
Assume $\overline{GE} \leq \overline{FG}$ so I_G is on $\overline{EM_{EF}}$. If not, just re-label E and F. By segment addition, $\overline{EI_G} + \overline{I_G M_{EF}} = \overline{EM_{EF}} = \overline{M_{EF}F} = \overline{I_G F} - \overline{I_G M_{EF}}$; thus, $2\overline{I_G M_{EF}} = \overline{I_G F} - \overline{EI_G}$.

$2\overline{I_G M_{EF}} = \overline{I_E F} - \overline{EI_F}$	Two tangents theorem
$= \overline{I_E F} - \overline{EI_F} + \overline{I_E G} - \overline{I_F G}$	Two tangents theorem
$= \overline{I_E F} + \overline{I_E G} - \overline{EI_F} - \overline{I_F G}$	Commutative property
$= \left(\overline{I_E F} + \overline{I_E G}\right) - \left(\overline{EI_F} + \overline{I_F G}\right)$	Associative property
$= \overline{FG} - \overline{GE}$	Segment addition ■

Incircle Theorem Corollary
Given $\overline{EFG}$ such that $\overline{EF} < \overline{FG} < \overline{GE}$, then $\overline{I_E M_{FG}} = \overline{I_G M_{EF}} + \overline{I_F M_{GE}}$.

This one is easy to get confused in your mind. Just remember that the segment on the middling side of the triangle is equal to the sum of the segments on the short and long sides of the triangle.

Problem 2.16 *Given $\overline{EFG}$ with I the incenter, drop a perpendicular from E onto $\overleftrightarrow{GI}$ with foot J and extend $\overrightarrow{EJ}$ to K on $\overrightarrow{GF}$. Prove that $2\overline{I_G M_{EF}} = \overline{FK}$.*

Proof
By the incircle theorem, $2\overline{I_G M_{EF}} = \left|\overline{FG} - \overline{GE}\right|$. By ASA, $\overline{GJK} \cong \overline{GJE}$, so $\overline{GK} = \overline{GE}$.

If $\overline{FG} = \overline{GE}$, then $2\overline{I_G M_{EF}} = 0$, but $K \equiv F$ by the center line theorem, so $\overline{FK} = 0$ too.
If $\overline{FG} < \overline{GE}$, then $2\overline{I_G M_{EF}} = \overline{GE} - \overline{FG} = \overline{GE} - \left(\overline{GK} - \overline{FK}\right) = \overline{GE} - \overline{GK} + \overline{FK} = \overline{FK}$.
If $\overline{FG} > \overline{GE}$, then $2\overline{I_G M_{EF}} = \overline{FG} - \overline{GE} = \overline{FK} + \overline{GK} - \overline{GE} = \overline{FK}$. ■

Problem 2.17
Given the base, how long must the legs of an isosceles triangle be if the incircle touches them at their trisection points?

Recall from the introduction that quadrilaterals are the union of two triangles adjacent on a side such that it is convex. They are congruent if their definitional triangles are pairwise congruent.

Side–Angle–Side–Angle–Side (SASAS) Theorem
Given three sides and the two angles between them, a quadrilateral is fully defined.

Proof
Given $\overline{EF}$ and $\angle F$ and $\overline{FG}$ and $\angle G$ and $\overline{GH}$, by SAS, $\overline{EFG}$ is fully defined. Quadrilaterals are convex, so E is inside $\angle G$ and, by the interior angle axiom, $\angle G = \angle FGE + \angle EGH$. By angle subtraction, $\angle EGH$ is fully defined. By SAS, $\overline{EGH}$ is fully defined. ■

Given sides and angles, there are four other quadrilateral congruence theorems: AAASS, AASAS, ASASA and ASSSS. There are more when given one or both diagonals. Proofs are left as exercises; in each case, you must prove that two triangles adjacent on a diagonal are both fully defined.

Problem 2.18
Given a segment $\overline{EF}$ that is cut by a line, ℓ, find a point G on ℓ such that ℓ bisects $\angle G$ in $\overline{EFG}$.

Solution
Let E' be the foot of the perpendicular dropped from E onto ℓ and let E'' be the reflection of E across ℓ. Analogously, define F' and F''. Assume $0 < \overline{FF'} < \overline{EE'}$; if it is not, then switch the labels. By SASAS, $\overline{EE'F'F''} \cong \overline{E''E'F'F}$, which holds the equality $\angle E = \angle E''$. $\overline{EE''G}$ is isosceles by the isosceles triangle theorem converse and ℓ is the perpendicular bisector of its base, $\overline{EE''}$. By the center line theorem, ℓ is also the apex angle bisector, so G is the desired point on ℓ. ■

Discussion
If ℓ is perpendicular to $\overleftrightarrow{EF}$, then $\overline{EE'F'F''}$ and $\overline{E''E'F'F}$ are degenerate. Two cases:

1. If ℓ is the bisector of $\overline{EF}$, then every point on ℓ that is not M_{EF} is a solution.
2. If ℓ is not the bisector of $\overline{EF}$, then there are no solutions.

If ℓ cuts $\overline{EF}$ at its midpoint but it is not perpendicular to $\overleftrightarrow{EF}$, then $\overline{EE'F'F''} \cong \overline{E''E'F'F}$ are Saccheri quadrilaterals with the same base; their summits extended never intersect. Since $\overrightarrow{EF''}$ and $\overrightarrow{E''F}$ are like this, then there is no solution. If either ℓ is perpendicular to $\overleftrightarrow{EF}$ or ℓ cuts $\overline{EF}$ at its midpoint, then the solution is not fully defined. Thus, we assume $\overline{FF'} < \overline{EE'}$, not $\overline{FF'} \leq \overline{EE'}$, and insist that $G := \overrightarrow{EF''} \cap \overrightarrow{E''F}$ exists and it is not on $\overline{EE''}$. ■

Tangential Quadrilateral Theorem I
A quadrilateral is tangential if and only if any three of its angle bisectors are concurrent.

Proof

Given $\overline{EFGH}$, assume that the angle bisectors of $\angle E, \angle F, \angle G$ are concurrent at I, then three uses of the angle bisector theorem on $\angle E, \angle F, \angle G$ prove I is equidistant to $\overline{EF}$ and $\overline{EH}$; also to $\overline{FE}$ and $\overline{FG}$; also to $\overline{GF}$ and $\overline{GH}$. By transitivity, it is equidistant to all the sides.

Assume that $\overline{EFGH}$ is tangential. By HL, $\overline{I_{HE}IE} \cong \overline{I_{EF}IE}$, so $\angle I_{HE}EI = \angle I_{EF}EI$. Thus, I is on the angle bisector of $\angle E$; analogously, I is on the angle bisectors of $\angle F, \angle G, \angle H$. ■

Pitot Theorem

In a tangential quadrilateral, the sums of each pair of opposite sides are equal.

Proof

Let $\overline{EFGH}$ be tangential and $I_{EF}, I_{FG}, I_{GH}, I_{HE}$ be the incircle's touching points. Then, $\overline{EF} + \overline{GH} = \overline{EI_{EF}} + \overline{I_{EF}F} + \overline{GI_{GH}} + \overline{I_{GH}H}$. By the two tangents theorem, $\overline{EI_{EF}} = \overline{EI_{HE}}$ and $\overline{FI_{EF}} = \overline{FI_{FG}}$ and $\overline{GI_{GH}} = \overline{GI_{FG}}$ and $\overline{HI_{GH}} = \overline{HI_{HE}}$. Thus, $\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$. ■

Lemma 2.3

A rhombus is tangential.

Pitot Theorem Converse

If the sums of each pair of opposite sides of a quadrilateral are equal, it is tangential.

Euclidean Proof (included for history-of-math students – only orange belts can do this)

Given $\overline{EFGH}$ with $\overline{EF} + \overline{GH} = \overline{EH} + \overline{GF}$, if all the sides are equal, the result by lemma 2.3. If not, then $\overline{EH} < \overline{EF}$, or re-label so this is true. For the equality to hold, $\overline{GH} < \overline{GF}$. Lay off $\overline{EH}$ on $\overrightarrow{EF}$ to J so $\overline{EH} = \overline{EJ}$. Lay off $\overline{GH}$ on $\overrightarrow{GF}$ to K so $\overline{GH} = \overline{GK}$. These sides are the bases of isosceles triangles $\overline{HJE}$, $\overline{JKF}$, $\overline{KHG}$. By the center line theorem, the mediators of the sides of $\overline{HJK}$ bisect $\angle E, \angle F, \angle G$; by the circumcenter theorem[43], they are concurrent. By tangential quadrilateral theorem I, $\overline{EFGH}$ is tangential. ■

Glagolev (1954) ended neutral geometry with the following proof, but was rebuked by Fetisov[44] (1954, pp. 26-27) for omitting L. 2.5. Pogorelov (1982) ended neutral geometry with L. 2.3, then began Euclidean geometry with the circumcenter theorem and then the above proof, which is standard in textbooks that do not start with neutral geometry as Euclid did in *The Elements*.

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[43] This is on page one of the orange-belt chapter; it is why this is a Euclidean proof. Look it up or come back later.

[44] At 15, in the small Russian town of Odojev, Fetisov taught himself French. He studied agriculture but pursued math as a hobby. At 37, in one sitting, he took the final exams of all 35 math classes taught by Moscow State University and became a Specialist of Abstract Math: like a B.S. For WWII, he taught geometry to artillery officers.

Lemma 2.4

Given $\overline{FG}$ with E and H on the same side of $\overleftrightarrow{FG}$, there exists a circle with center P that touches $\overrightarrow{FE}$, $\overline{FG}$, $\overrightarrow{GH}$ at P_{EF}, P_{FG}, P_{GH}, respectively.

Proof

Since $\frac{1}{2}\angle F + \frac{1}{2}\angle G < \sigma$, the bisectors of $\angle F$ and $\angle G$ intersect at P. By the angle bisector theorem, P is equidistant from $\overrightarrow{FE}$, $\overline{FG}$ *and* $\overrightarrow{GH}$. ■

Lemma 2.5

Given a quadrilateral with the sums of each pair of opposite sides equal, by L. 2.4, the vertices can be sequentially labeled E, F, G, H so a circle exists that touches at P_{EF}, P_{FG}, P_{GH}. E inside $\overline{FP_{EF}}$ and H inside $\overline{GP_{GH}}$ are not both true.

Proof

Assume that E is inside $\overline{FP_{EF}}$ and that H is inside $\overline{GP_{GH}}$.

$\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$	Given
$\overline{FP_{EF}} - \overline{EP_{EF}} + \overline{GP_{GH}} - \overline{HP_{GH}} = \overline{HE} + \overline{FP_{FG}} + \overline{GP_{FG}}$	Expand
$-\overline{EP_{EF}} - \overline{HP_{GH}} = \overline{HE}$	Two tangents theorem

$\overline{HE} + \overline{EP_{EF}} + \overline{HP_{GH}} = 0$ is impossible, so E is outside $\overline{FP_{EF}}$ or H is outside $\overline{GP_{GH}}$. ■

Pitot Theorem Converse

If the sums of each pair of opposite sides of a quadrilateral are equal, it is tangential.

Neutral Geometry Proof

Given $\overline{EFGH}$ with $\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$, by lemma 2.4, it is possible to construct a circle tangent to $\overline{FG}$ and $\overrightarrow{FE}$ and $\overrightarrow{GH}$. Suppose that it is not tangent to $\overline{HE}$. By lemma 2.5, it is possible to draw a tangent to this circle from E to a point K on $\overrightarrow{GH}$. $\overline{EFGK}$ is tangential. The trichotomy is $K \equiv H$, K is inside $\overline{GH}$, or K is on $\overrightarrow{GH}$ past H. Assume K is inside $\overline{GH}$.

$\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$	Given
$\overline{EF} + \overline{GK} = \overline{FG} + \overline{KE}$	Pitot theorem using $\overline{EFGK}$
$\overline{GH} - \overline{GK} = \overline{HE} - \overline{KE}$	Subtract one equation from the other
$\overline{KH} = \overline{HE} - \overline{KE}$	Subtract $\overline{GK}$ from $\overline{GH}$
$\overline{KH} + \overline{KE} = \overline{HE}$	Add $\overline{KE}$ to both sides

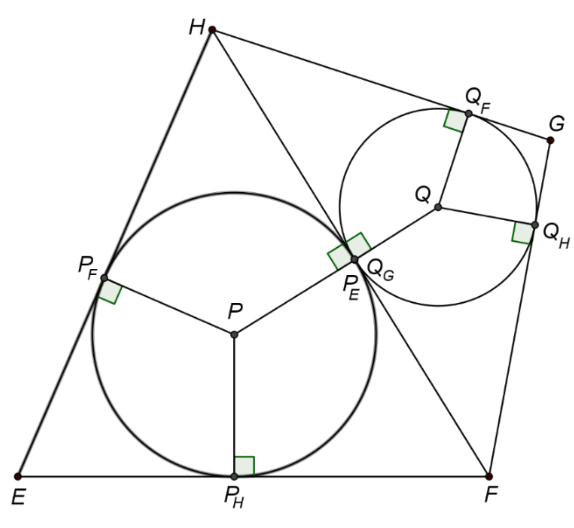
This contradicts the triangle inequality theorem in $\overline{KHE}$, so K is not inside $\overline{GH}$. Analogously, K is not on $\overrightarrow{GH}$ past H. Thus, $K \equiv H$ and the circle is tangent to $\overline{HE}$. ■

Pedal triangle vertices are the feet of perpendiculars dropped on triangle sides or their extension from an arbitrary point called the **pedal point**. Orange belts will prove the orthocenter theorem, that the altitudes are concurrent at a point called the orthocenter, which makes the **orthic triangle** a pedal triangle. If the pedal point is the circumcenter, then the pedal triangle is the medial triangle, its sides are **mid-segments**, and its vertices are the midpoints of its **parent triangle**. Without the parallel postulate, the **circumcircle** may not exist, but it does for equilateral triangles, so it is a yellow-belt term; medial triangles exist now, but only become important later.

If the pedal point is the incenter, then the incircle touches the triangle at the vertices of the **contact triangle**, which is pedal. The incircle exists and is unique for every triangle, in contrast to tangential quadrilaterals, which are not fully defined by their sides. We have existence when $a + c = b + d$, but we never have uniqueness because the length of the diagonal can be any x such that $0 < x < a + b$. Choose one and then, by SSS, draw in c and d.

Before proving the following theorem, let us state what a quadrilateral is because our definition differs from the rather vacuous definition (four-sided figure) that is given in other textbooks. A quadrilateral is the union of two triangles adjacent on a side. In *Geometry–Do*, all quadrilaterals are convex; we ignore four-sided concave figures. The definitional diagonal is the adjacent side.

Next, let $\overline{EFGH}$ be the union of $\overline{EFH}$ and $\overline{FGH}$. Let P_E, P_F, P_H be the pedal triangle vertices of the incenter of $\overline{EFH}$. Let Q_F, Q_G, Q_H be the pedal triangle vertices of the incenter of $\overline{FGH}$. The subscript refers to the opposite vertex; *e.g.,* P_E is on side $\overline{FH}$ of triangle $\overline{EFH}$. Calling "Two tangents!" every time we substitute equal segments is tedious, so this will be left tacit.



A pedal point is denoted P and, if there is a second one, it is Q; with subscripts, these letters denote the pedal triangle vertices. The incenter, I, is a pedal point; the notation is compatible.

Tangential Quadrilateral Theorem II
The incircles of a quadrilateral's two triangles are tangent if and only if it is tangential.

Part One
If a quadrilateral is tangential, then the incircles of its two triangles are tangent.

We hope to prove that P_E and Q_G are the same point; that is, the length of $\overline{P_EQ_G}$ is zero. This is a trichotomy with the other two alternatives being $\overline{FQ_G} < \overline{FP_E}$ or $\overline{FP_E} < \overline{FQ_G}$.

Proof
$\overline{FP_E} = \overline{EF} - \overline{EP_H}$ and $\overline{FQ_G} = \overline{FG} - \overline{GQ_H}$; thus,
$\overline{P_EQ_G} = \overline{FP_E} - \overline{FQ_G} = \overline{EF} - \overline{EP_H} - \overline{FG} + \overline{GQ_H}$. Here we are assuming $\overline{FQ_G} < \overline{FP_E}$.

$\overline{HQ_G} = \overline{HG} - \overline{GQ_F}$ and $\overline{HP_E} = \overline{HE} - \overline{EP_F}$; thus,
$\overline{P_EQ_G} = \overline{HQ_G} - \overline{HP_E} = \overline{GH} - \overline{GQ_F} - \overline{HE} + \overline{EP_F}$.

$2\overline{P_EQ_G} = \overline{EF} + \overline{GH} - \left(\overline{FG} + \overline{HE}\right)$ by adding the two equations above.
But $\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$ by the Pitot theorem, so $\overline{FQ_G} < \overline{FP_E}$ cannot be true.
Analogously, $\overline{FP_E} < \overline{FQ_G}$ cannot be true, so $\overline{P_EQ_G} = 0$. •

Part Two
If the incircles of a quadrilateral's two triangles are tangent, then it is tangential.

Here we assume that P_E and Q_G are the same point, so we will give it one name, T. We hope to prove that $\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$ so that we can then invoke the Pitot theorem converse.

Proof

$$
\begin{aligned}
\overline{EF} + \overline{GH} &= \overline{EP_H} + \overline{P_HF} + \overline{GQ_F} + \overline{Q_FH} \\
&= \overline{EP_F} + \overline{FT} + \overline{GQ_H} + \overline{HT} \\
&= \overline{EP_F} + \overline{FQ_H} + \overline{GQ_H} + \overline{HP_F} = \overline{HE} + \overline{FG}
\end{aligned}
$$

Thus, by the Pitot theorem converse, $\overline{EFGH}$ is tangential. • ■

Tangential Quadrilateral Theorem III
Let P_F and P_H be pedal triangle vertices of $\overline{EFH}$, Q_G and Q_E be pedal triangle vertices of $\overline{EFG}$, R_H and R_F be pedal triangle vertices of $\overline{GHF}$ and S_E and S_G be pedal triangle vertices of $\overline{GHE}$. Then $\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$ if and only if $\overline{P_HQ_G} + \overline{R_FS_E} = \overline{Q_ER_H} + \overline{S_GP_F}$.

Expand each addend of $\overline{EF} + \overline{GH} = \overline{FG} + \overline{HE}$ into the sum of three segments and then reassemble into $\overline{P_H Q_G} + \overline{R_F S_E} = \overline{Q_E R_H} + \overline{S_G P_F}$; or *vice versa*. This is left as an exercise.

Шарнирная теорема means "hinge theorem" in Russian, but this is not a name that Russian geometers would recognize. It was proven by Euclid (*The Elements*, Book I, Prop. 24, 25), but he never cited it in a subsequent proof, which probably explains why the Russians mostly ignore this result. It is a staple of Western high-school geometry, but it is just added to that vast pile of stuff that high-school students must memorize, though they know that they will never use any of it.

But all that is about to change. The hinge theorem can be cited in trichotomy proofs!

The **bi-medial** – the intersection of quadrilateral diagonals – is usually denoted T. A **kite** is the union of congruent triangles such that uncommon sides that are equal are also adjacent.

Tangential Quadrilateral Theorem IV
If a quadrilateral is tangential and the midpoint of one diagonal is its bi-medial, then it is a kite.

Let $\overline{EFGH}$ be tangential and $M_{FH} \equiv T$. Label $w = \overline{EI_{HE}} = \overline{EI_{EF}}$ and $x = \overline{FI_{EF}} = \overline{FI_{FG}}$ and $y = \overline{GI_{FG}} = \overline{GI_{GH}}$ and $z = \overline{HI_{GH}} = \overline{HI_{HE}}$; these equalities are true by the two tangents theorem.

The proof would be easy if we knew $\overline{EFGH}$ to be **orthodiagonal**, so $\angle T = \rho$. Then, $\overline{ETF} \cong \overline{ETH}$ and $\overline{GTF} \cong \overline{GTH}$ by SAS, which hold the equalities $\overline{EF} = \overline{EH}$ and $\overline{GF} = \overline{GH}$, respectively. A direct proof that $\angle T = \rho$ eludes me, but there are only two other alternatives: $\angle ETF < \angle ETH$ or $\angle ETH < \angle ETF$. If these alternatives are contradictory, then $\angle T = \rho$ by trichotomy.

Proof
Assume that $\angle ETF < \angle ETH$. $\overline{ETF}$ and $\overline{ETH}$ hold the inequality $\overline{EF} < \overline{EH}$ by the hinge theorem. Subtract w from both sides to get $x < z$. By the vertical angles theorem, $\angle GTH = \angle ETF$ and $\angle GTF = \angle ETH$; thus, $\angle GTH < \angle GTF$. $\overline{GTH}$ and $\overline{GTF}$ hold the inequality $\overline{GH} < \overline{GF}$ by the hinge theorem. Subtract y from both sides to get $z < x$. This is a contradiction, so $\angle ETF \nless \angle ETH$. Analogously, $\angle ETH \nless \angle ETF$. ■

We will conclude yellow belt with a discussion of isosceles triangles, which are a principal topic of neutral geometry. But first we will prove an important theorem that serves as a bridge between yellow belt for the Average Joe and advanced yellow belt for aspiring mathematicians.

Did everybody get the white-belt exit exam question about Saccheri quadrilaterals?

Mid–Segment and Mediator Theorem

The mid-segment of a triangle's sides is perpendicular to the mediator of its base.

Proof

Given $\overline{EFG}$, let E'', F'', G'' be the feet of perpendiculars dropped on $\overleftrightarrow{M_{GE}M_{FG}}$ from E, F, G, respectively. By AAS, $\overline{E''M_{GE}E} \cong \overline{G''M_{GE}G}$ and $\overline{F''M_{FG}F} \cong \overline{G''M_{FG}G}$; thus, $\overline{E''E} = \overline{G''G}$ and $\overline{F''F} = \overline{G''G}$. By transitivity, $\overline{E''E} = \overline{F''F}$, so $\overline{E''F''FE}$ is a Saccheri quadrilateral. By Saccheri theorem I, $\overleftrightarrow{M_{E''F''}M_{EF}} \perp \overleftrightarrow{E''F''}$; but, $\overleftrightarrow{E''F''} \equiv \overleftrightarrow{M_{GE}M_{FG}}$. ■

It is an easy corollary of ASA that isosceles triangles have two angle bisectors equal; while not among Euclid's propositions, it must surely have been known to him. Is the converse also this easy? No. Geometers waited over 2000 years for the converse, which is known as the Steiner-Lehmus theorem. Here is not the classic proof of Jakob Steiner, but a better one because it does not invoke the parallel postulate. It is made possible by Pasch's axiom, which was introduced by Moritz Pasch in 1882, about forty years after Lehmus proposed and Steiner solved the problem.

Steiner–Lehmus Theorem

If a triangle has two angle bisectors equal, then it is isosceles.

Modern Proof

Given $\overline{EFG}$ with the angle bisectors $\overline{FF^*} = \overline{EE^*}$, assume that $\overline{GE} < \overline{GF}$. By the greater angle theorem, $\angle F < \angle E$, and their halves, $\angle F^*FG < \angle E^*EG$. By C. 1.5, replicate $\angle F^*FG$ with one ray $\overrightarrow{EE^*}$ and the other inside $\angle E^*EG$; let J be its intersection with $\overline{GF}$. From $\angle F^*FG < \angle E^*EG$ and insideness, $\angle EFJ < \angle FEJ$. By the greater side theorem, $\overline{EJ} < \overline{FJ}$, so one can lay off $\overline{FM} = \overline{EJ}$ with M inside $\overline{FJ}$. By SAS, $\overline{JEE^*} \cong \overline{MFF^*}$, which holds the equality $\angle EJE^* = \angle FMF^*$. Applying Pasch's axiom to the line $\overleftrightarrow{EJ}$ and the triangle $\overline{MGF^*}$, there exists a point N that is on $\overleftrightarrow{EJ}$ and between F^* and M. In the triangle $\overline{NJM}$, the interior angle $\angle NJM$ equals the exterior angle $\angle NMF$ because $\angle EJE^* = \angle FMF^*$. This contradicts the exterior angle inequality theorem and so $\overline{GE} = \overline{GF}$. ■

For pedagogic purposes and because of its importance to the history of geometry, the classic proof of the Swiss mathematician Jakob Steiner is given at the end of the book in an appendix.

We can now state this important theorem as a bi-conditional. And we do so here, in yellow belt!

Isosceles Angle Bisectors Theorem

Two angle bisectors are equal if and only if the triangle is isosceles.

The name and wording of this theorem may remind you of one we proved immediately after AAS.

Isosceles Altitudes Theorem
Two altitudes are equal if and only if the triangle is isosceles.

There seems to be a pattern developing. What about the medians? Problem 1.4.1 asked for proof that, if a triangle is isosceles, then two medians are equal. How many of you were left with a hole in your heart when we concluded white belt without proving the converse?

Isosceles Medians Theorem
Two medians are equal if and only if the triangle is isosceles.

> ***Proof***
> If $\overline{EFG}$ is isosceles so $\overline{EG} = \overline{FG}$, then, by SAS, $\overline{EGM_{FG}} \cong \overline{FGM_{GE}}$. Thus, $\overline{EM_{FG}} = \overline{FM_{GE}}$.

Proof of the converse would be easy if we knew $\angle EFM_{GE} = \angle FEM_{FG}$. Then, $\overline{EFM_{GE}} \cong \overline{FEM_{FG}}$ by SAS, which holds the equality $\overline{EM_{GE}} = \overline{FM_{FG}}$; by doubling, $\overline{EG} = \overline{FG}$. A direct proof that $\angle EFM_{GE} = \angle FEM_{FG}$ eludes me, but there are only two other alternatives: $\angle EFM_{GE} < \angle FEM_{FG}$ or $\angle FEM_{FG} < \angle EFM_{GE}$. If these alternatives are contradictory, then $\angle EFM_{GE} = \angle FEM_{FG}$ by trichotomy. This is the same strategy that we used to prove tangential quadrilateral theorem IV.

> Given $\overline{EFG}$ such that $\overline{EM_{FG}} = \overline{FM_{GE}}$, extend $\overrightarrow{EM_{FG}}$ to P so $\overline{EM_{FG}} = \overline{PM_{FG}}$ and extend $\overrightarrow{FM_{GE}}$ to Q so $\overline{FM_{GE}} = \overline{QM_{GE}}$. By SAS, $\overline{EM_{FG}G} \cong \overline{PM_{FG}F}$ and $\overline{FM_{GE}G} \cong \overline{QM_{GE}E}$, which holds the equalities $\overline{EG} = \overline{PF}$ and $\overline{FG} = \overline{QE}$, respectively.
>
> Assume that $\angle EFM_{GE} < \angle FEM_{FG}$. $\overline{EFQ}$ and $\overline{FEP}$ hold the inequality $\overline{EQ} < \overline{FP}$ by the hinge theorem. $\overline{EFM_{GE}}$ and $\overline{FEM_{FG}}$ hold the inequality $\overline{EM_{GE}} < \overline{FM_{FG}}$ by the hinge theorem. By doubling, $\overline{EG} < \overline{FG}$; by substitution, $\overline{PF} < \overline{QE}$. This is a contradiction, so $\angle EFM_{GE} \nless \angle FEM_{FG}$. Analogously, $\angle FEM_{FG} \nless \angle EFM_{GE}$. ■

For the experts[45] who know of the two-to-one medial point theorem, a Euclidean result that assumes the parallel postulate, this theorem may seem trivial.[46] But proving this here in yellow belt makes for a much more powerful result because it makes it true in hyperbolic geometry as well as in Euclidean geometry. Isosceles triangles are very important in hyperbolic geometry! The Lobachevskians would be lost without knowing the conditions equivalent to being isosceles.

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[45] What are you doing spying on the beginner chapter?
[46] $\overline{ECM_{GE}} \cong \overline{FCM_{FG}}$ by SAS, which holds the equality $\overline{EM_{GE}} = \overline{FM_{FG}}$. By doubling, $\overline{EG} = \overline{FG}$.

Let us summarize the conditions known to be equivalent to a triangle being isosceles:

Two interior angles are equal	Two altitudes are equal
Two angle bisectors are equal	Two medians are equal

Problem 2.19
Given an isosceles right triangle, can you prove that the base angles are each half of a right angle?

No, you cannot prove this! Yellow belts do not have the angle sum theorem (the interior angles of a triangle sum to one straight angle) because its proof cites the parallel postulate. In hyperbolic geometry, the interior angles of a triangle sum to less than a straight angle; $\alpha + \beta + \gamma < \sigma$.

A **rectangle** is a quadrilateral with equal angles and a **square** is a rectangle with equal sides.

Construction 2.4 ***(Euclid, Book IV, Prop. 15)***
Inscribe a regular (equilateral and equiangular) hexagon in a given circle.

Euclidean Solution
Draw circles equal to the given circle with centers **diametrically opposed** in it. The new circles' centers and their intersections with the given circle are the hexagon vertices. ■

Common Core sometimes tacks this construction onto their annoying fill-in-the-bubble exams.

Neutral Geometry Solution
Draw an equilateral triangle and find its incenter, I. By the incenter and circumcenter theorem, the angle bisectors extended mediate the opposite sides. Thus, there are six triangles congruent by AAS, so the six angles around I are equal. Draw a circle around I of the given radius. By SAS, the six rays cut it at the vertices of a regular hexagon. ■

Sometimes *Common Core* asks for a square in a circle, which is easy – a diameter and its mediator cut the circle at its vertices. By SAS, it has equal sides and equal angles. But they may be acute!

Right Rectangle	A rectangle with right angles
Right Square	A right rectangle with equal sides

Yellow belts are responsible for knowing these definitions only so they can contrast what they are doing with what orange belts will be doing after the parallel postulate is introduced. Yellow belts *cannot* construct either of these figures. When René Descartes invented Cartesian coordinates, he did not just construct a grid of squares, he constructed a grid of *right* squares.

Cartesian coordinates are only useful if the scale is stated so the problem can be applied to some object in the real world. But scaling a geometric figure up or down to represent a real-world object can only be done with the triangle similarity theorem, which states that, if the angles of two triangles are pairwise equal, then then their sides are proportional. But this theorem cannot be proven without the parallel postulate.

Suppose I say, "Given a triangle with base 40.2 cm and base angles $45°$ and $51°$, what is the length of the median to the base?" This is an easy application of the ASA theorem. You draw the base to the given length, use a protractor to construct the angles and extend their rays to intersect at the apex. You use C. 1.2 to bisect the base and then you use your centimeter ruler to measure from the base midpoint to the apex. It is 22.3 cm!

Now suppose that I try to make this problem realistic by saying, "I have a mortar and wish to shell an enemy anti-aircraft gun. But I cannot measure the distance directly because it is on the other side of an office building and, even if I could get out there with a measuring tape, they would shoot me. So, I send two soldiers 201 m in each direction on my street and have them measure the angle to the enemy gun. They report angles of $45°$ and $51°$. What is the range for my shot?"

Without the parallel postulate, we cannot answer. Just because the base of the triangle on the paper and the triangle in real life have a ratio of one to a thousand is no assurance that the legs or the median to the base are in this ratio. Ballistics is not just different in a hyperbolic world, it does not exist, at least on a large scale. It might be approximately Euclidean on the scale of small arms, but artillery would be pointless because there would be no means of locating the enemy.

Descartes is known as the father of modern philosophy; he is a staple of college philosophy courses. Even in high school, most students are familiar with his dictum, *Cogito, ergo sum*; or, *I think, therefore I am*. Descartes wanted to clear away all tacit assumptions – *A noble endeavor, which I share!* – and he concluded that "I exist" is the only thing that any of us can really be sure of. From this one axiom, he then set out to prove all knowledge. With all due respect, Descartes failed. His invention of Cartesian coordinates tacitly assumes the parallel postulate. His *Fifth Meditation* "proves" the existence of God by stating that existence is in the definition of a perfect being in the same way that the angle sum theorem is in the definition of a triangle. It is not.

Common Core tacitly assumes the parallel postulate. We will play a trick on American teachers who skip ahead to the yellow belt exit exam to check that they know this material without reading the chapter. In problems #4 and #5, I define geometric figures by their Cartesian coordinates, so we are assuming that the world is approximately Euclidean for triangles a few hundred meters across. The American teachers' insistence on turning easy geometry problems into difficult algebra problems will be hilarious as they futilely throw square roots in every direction.

Yellow Belt Exit Exam

1. ***Conway Problem*** *At each vertex, extend the sides of a triangle out by a distance equal to the opposite side. Prove that the six endpoints are* ***concyclic*** *and find the circle center.*

2. *A triangle has sides of* 13 *cm,* 14 *cm and* 15 *cm. Where does the incircle touch the sides?*

3. *Prove that, if the apex angle bisector and the base mediator do not coincide, then they intersect outside the triangle.*

4. *Find the circumcenter of a triangle with vertices at* $(-3.8, -0.6)$, $(12.7, 0.4)$ *and* $(1, -27)$.

5. *You bought a quadrilateral farm with vertices at* $(0, 0)$, $(408, 0)$, $(288, 315)$ *and* $(68, 285)$ *meters. You wish to install center pivot irrigation. Where is the center?*

6. *Given* P *interior to* $\overline{EFG}$, *let* $E'' := \overrightarrow{EP} \cap \overline{FG}$ *and* $F'' := \overrightarrow{FP} \cap \overline{EG}$ *and* $G'' := \overrightarrow{GP} \cap \overline{EF}$. *Prove that, if any two of* $\overline{EG''PF''}$ *or* $\overline{FE''PG''}$ *or* $\overline{GF''PE''}$ *are tangential quadrilaterals, then the third quadrilateral is also tangential.*

7. *Let* $\overline{EFGH}$ *be tangential with* $\angle GHE + \angle HEF < \sigma$ *and* I *the incenter.* J *is on* $\overleftrightarrow{I_{EF}I_{GH}}$ *such that* $\overline{HJ} = \overline{HI_{GH}}$. *Prove that* $\overrightarrow{EI}$ *bisects* $\overline{I_{EF}J}$.

8. *Look up* ***external tangents*** *and then prove this: If two circles touch, then the quadrilateral whose vertices are the touching points of their external tangents is tangential.*

9. *Given two sides, an angle opposite one of them and the fact that the angle opposite the other side is acute or that it is obtuse, prove that the triangle is fully defined.*

10. *Given lines* $\overleftrightarrow{EF}$ *and* $\overleftrightarrow{GH}$ *with* F *and* H *on opposite sides of* $\overleftrightarrow{EG}$, *if* $\angle FEG = \angle HGE$, *prove that* $\overleftrightarrow{EF}$ *and* $\overleftrightarrow{GH}$ *do not intersect.*

11. *Given* $\overline{EFG}$, *let* E'', F'', G'' *be the feet of perpendiculars dropped on* $\overleftrightarrow{M_{GE}M_{FG}}$ *from* E, F, G, *respectively. Prove that* $\angle E''EF + \angle F''FE$ *is the angle sum of* $\overline{EFG}$.

12. *Construct an angle that is one third of a straight angle. Can we call this angle* φ*?*

13. *A soldier must sweep an equilaterally triangular field using a landmine detector that has a radius of half the triangle's altitude. If he starts at one vertex, what is the shortest path?*

Practice Problems

2.20 Construct a right triangle given

1. *a leg and the hypotenuse.*
2. *the hypotenuse and one acute angle.*
3. *the sum and the difference of the legs.*

2.21 Construct a triangle given

1. *the base, the apex altitude and one leg.*
2. *the base and the two base angles.*
3. *the base, a base angle and the median to the leg that defines this angle.*
4. *the base, the median to the base and one leg.*
5. *the apex angle, its angle bisector (the length to its infoot) and a leg.*

2.22 Construct two equal circles whose common chord equals their given radius.

2.23 Construct two equal circles given their common chord and the distance between centers.

2.24 Given that $M := \overline{EF} \cap \overline{GH}$ *and* $\overrightarrow{MJ}$ *bisects* $\angle FMG$,

1. *Prove that* $\overrightarrow{JM}$ *bisects* $\angle EMH$.
2. *If* $\overrightarrow{MK}$ *bisects* $\angle EMH$, *prove that* J, K *and* M *are collinear.*

2.25 Given $\overline{EFGH}$ *tangential,* $\overline{GH} = 2\overline{FG} = 4\overline{EF}$, *how many times can* $\overline{EF}$ *be laid off in* $\overline{HE}$?

2.26 Solve problem 2.11 if the angle goes between the centers; that is, its rays cut $\overline{O_1O_2}$.

2.27 Let ω_1 *and* ω_2 *be concentric circles with* ω_1 *inside* ω_2 *and center* O. *If* E, F *are on* ω_1 *and* G, H *are on* ω_2 *and* $\angle OEG = \angle OFH$, *prove that* $\overline{OEG} \cong \overline{OFH}$.

2.28 Let $\overline{EFG}$ *be isosceles with* $\overline{EG} = \overline{FG}$ *and* E' *and* F' *be the feet of altitudes to* $\overleftrightarrow{FG}$ *and* $\overleftrightarrow{EG}$, *respectively. Prove that the line from* G *through their intersection bisects* $\angle G$.

2.29 Let $\overline{EFG}$ *be isosceles with* $\overline{EG} = \overline{FG}$ *and let* P *be a point in the interior of* $\overline{EFG}$ *such that* $\angle EPG = \angle FPG$. *Prove that* P *is on the median to* $\overline{EF}$.

2.30 Let P_1, P_2, P_3, P_4, P_5 *be distinct points on a circle such that* $\overline{P_1P_2} = \overline{P_2P_3} = \overline{P_3P_4} = \overline{P_4P_5}$. *Prove that* $\overline{P_1P_4} = \overline{P_2P_5}$.

2.31 *You own a farm on one side of a straight paved highway and with your barn some distance from the highway. Build a curved driveway of a constant turning radius that meets the highway at the edge of your property line and is tangent to it. Describe how. This construction may not be possible in hyperbolic geometry. Why not?*

2.32 *Given $\overline{EFG}$ with $\overline{EG} = \overline{FG}$, extend $\overrightarrow{GE}$ to J and $\overrightarrow{GF}$ to K such that $\angle GJF = \angle GKE$. Prove that $\overline{JGF} \cong \overline{KGE}$.*

2.33 *Given $\overline{EF}$ with J and K on the mediator such that $\overline{EJ} = \overline{FK}$. Prove that $\overline{EM_{EF}J} \cong \overline{FM_{EF}K}$.*

2.34 *Given $\overline{EFG}$ with $\angle EFG = \angle FEG$, prove that $\overline{EM_{EF}G} \cong \overline{FM_{EF}G}$.*

2.35 *Given $\overline{EF}$ with $\angle EJM_{EF} = \angle FKM_{EF}$ and with J, M_{EF}, K collinear and in that order, prove that $\overline{EJM_{EF}} \cong \overline{FKM_{EF}}$.*

2.36 *Given $\overline{EF}$ and a line that cuts it at M_{EF} but is not perpendicular to $\overleftrightarrow{EF}$, raise perpendiculars from E and F to intersect the line at J and K, respectively. Prove that $\overline{EJM_{EF}} \cong \overline{FKM_{EF}}$.*

2.37 *Given $\overline{EFGH}$ with $\angle E = \angle F = \rho$ and $\overline{EG} = \overline{FH}$, prove that $\overline{EFGH}$ is a Saccheri quadrilateral.*

2.38 *Kite is an orange-belt term; look it up. (1) Construct a kite; (2) Without the Pitot theorem converse, prove that kites are tangential; (3) Prove that one diagonal mediates the other.*

2.39 *Some books include among the perpendicular length theorem corollaries that, if a line is perpendicular to one of two parallels, it is perpendicular to the other too. Why don't we? Is it also a corollary that tangents to a circle at the endpoints of a diameter never meet?*

2.40 *Let $\overline{EF}$ be a diameter and O the circle center. If P is on $\overline{OE}$, prove that $\overline{PE} \leq \overline{PG} \leq \overline{PF}$ for any G on the circle. If P is on $\overrightarrow{OE}$ past E, does this inequality still hold?*

2.41 *Given $\overline{EFG}$ isosceles and $\angle G$ obtuse, let E' and F' be the feet of perpendiculars dropped on $\overrightarrow{FG}$ and $\overrightarrow{EG}$, respectively. Let $G'' := \overrightarrow{EE'} \cap \overrightarrow{FF'}$. Prove that $\overline{EFG''}$ is isosceles.*

2.42 *Let E, F, G be on an O-circle such that $\overline{EF} < \overline{FG}$, but O is not on $\overline{FG}$. Prove that $\angle OM_{EF}M_{FG} < \angle OM_{FG}M_{EF}$.*

2.43 *Given two intersecting lines and a point, G, that is not on either of them, find E on one line and F on the other line so the lines are angle bisectors of $\overline{EFG}$.*

Introductory Geometry the Year Before *Geometry–Do* Is Taught[47]

Prerequisites. Introductory geometry does not require quadratic equations. Neither the quadratic formula, the distance formula, the midpoint formula, nor square roots are needed. But geometers *are* expected to have a working knowledge of solving linear equations by isolating the variable and, while they do not need a general knowledge of linear systems, they should be able to solve triangular systems by substitution. This is an example of introductory algebra:

$3x - 4 = 5$ Solution: $x = 3$	$2x - y = 5$ Solution: $2y = 6$ $x = 4$ and $y = 3$	$y = -2x + 4$ x and y intercepts: $x = 2$ and $y = 4$

Geometers do not initially have multiplication, but they can bisect and trisect segments. Problems with halves or thirds that have solutions with denominators of $2, 3, 4, 6$ are useful. This should be taught in Level VII, but the introductory geometry textbook that I am recommending for Level VIII is quite basic and it is possible to teach it concurrently with introductory algebra. Introductory algebra is needed for Level IX geometry, and mastery is needed in Level X.

Textbook. For 2000 years, *The Elements* was the standard and only geometry textbook. It was what Newton relied on to prove all the theorems in his *Principia*.[48] However, *The Elements* is difficult for modern math students to read for the same reason that *Beowulf* is difficult for literature students to read – they are just so *old!* Thomas Heath did a good job translating *The Elements* into English, but it is still difficult reading for teachers and wholly unsuited for students.

In the early 20th century, the Englishmen Hall and Stevens wrote *A School Geometry*, which proves the most important propositions while relegating many to homework problems. In 1918, this was the middle- and high-school textbook in Western Canada. In 2017, it was reprinted in Delhi and is now sold as a college textbook for aspiring teachers with no mention of its origins, probably because Western Canada was mostly wilderness and vast cattle ranches in 1918 – much as it still is – and "progressives," as they like to call themselves, do not want to admit that teenage cowboys from a hundred years ago knew far more geometry than most American and Canadian high-school teachers do today. Being progressive only makes sense if we are actually advancing!

I recommend *A School Geometry* for Level VIII Indians who will study *Geometry–Do* in Levels IX and X.[49] To help geometry teachers coordinate their curriculums, I will describe using *A School*

[47] A more detailed discussion is here: www.axiomaticeconomics.com/Teachers_Manual_School_Geometry.pdf

[48] A somber man, the only time he is known to have laughed was when he loaned a copy of *The Elements* to a friend, who asked if it was really necessary that he read it. Newton laughed in his face. It was the stupidest question, *ever!*

[49] Despite the bold title, *Elementary Geometry for College Students* by Alexander and Koeberlein is suitable for Level VIII Indians or 10th grade Americans. It is three times the price, but it is modern; Hall and Stevens are old fashioned. *Introduction to Geometry* by Rusczyk is not suitable because he is just teaching remedial algebra with illustrations.

Geometry to prepare for *Geometry–Do*. Also, home-school Americans who go on to a Catholic high school that uses *Geometry–Do* may be asked to teach their younger siblings; they too need this. *Common Core* students can read it on their own to prepare for attending a private school that uses *Geometry–Do*, or to at least learn the bare minimum that college professors expect.

Old-time geometers use the term "produce," while modern geometers say "extend." A "straight line" is a line; all lines are straight. Sometimes they say line but mean a line segment. Euclid used the term "proposition;" modern geometers say "theorem." Theorem 3. [Euclid I. 15] means that what follows is the third theorem in *A School Geometry* and the 15th proposition in Book I of *The Elements*. Otherwise, the English is easily understood by modern Level VIII Indians and by American home schoolers without any explanation beyond what the authors provide on p. 6.

I recommend that the teacher write in the margins the modern names of theorems to help students transition to *Geometry–Do*.[50] Because the difficulty level is uneven, it should be taught in this order to bring the easy circle theorems forward and leave the difficult ones for later.

1, 2	Supplementarity	31	Diameter and Chord Theorem
3	Vertical Angles Theorem	32	Circumcenter Theorem Corollary
4	SAS (Side–Angle–Side)	33	Diameter and Chord Theorem Corollary #2
5	Isosceles Triangle Theorem	34	Equal Chords Theorem
6	Isosceles Triangle Theorem Converse	46	Tangent Theorem
7	SSS (Side–Side–Side)	47	Two Tangents Theorem
8	Exterior Angle Inequality Theorem	48	Common Point Theorem
9	Greater Angle Theorem		**Optional**
10	Greater Side Theorem	19	Hinge Theorem
11	Triangle Inequality Theorem	20	Equal Segments on Parallels Theorem
12	Perpendicular Length Theorem	21	Misc. Parallelogram Theorems
13	Transversal Lemma	22	Two Transversals Theorem
14	Transversal Theorem	24	Parallelogram Area Theorem
15	Transitivity of Parallels	25	Triangle Area Theorem Corollary #3
16	Angle Sum Theorem	26	Triangle Area Theorem
17	ASA and AAS (Two Angles and a Side)	29	Pythagorean Theorem
18	HL (Hypotenuse–Leg)	30	Pythagorean Theorem Converse

Teachers should work through chapters 1, 2, 3 and 4.[51] Bring the easy circle theorems ahead of parallelograms by doing theorems 31, 32, 33, 34, 46, 47 and 48. Get compasses and straightedges and do the constructions (called problems) 1, 2, 3, 4, 5, 6, 8, 9, 10, 14, 15, 20, 21, 22, 25, 26 and 27 using the first methods given. 26 and 27 are needed for *Geometry–Do*; do not omit them!

[50] Go through the table of contents and write the chapter numbers in to reflect the modern division into chapters.
[51] Theorem 18; 19 is optional. Beware! The PDF files and the Delhi edition differ in their circle theorem numbers!

Off–the–Grid Cabins as an Application of Geometry

Suppose you are building an $8' \times 12'$ cabin that will rely entirely on solar power. The roof should be asymmetric, with the southern face of a mild pitch to support the solar panels while the northern face is more steeply pitched. Also, for a short $8'$ span, asymmetric trusses are stronger.

The strong roof truss. For an asymmetrical roof, we will modify the Fink truss so the attic is a half equilateral triangle. Bisect the ceiling joist and construct an equilateral triangle on the northern side. The rafters rest on the southern wall and on the triangle apex. A beam from the foot of the triangle meets the rafter at a right angle.

The hollow roof truss. The ceiling joist and the rafters are the same, but the support beams are removed to house a round water tank. Water tanks can be quite heavy so, even if there were not a need for a large southern roof face, it is a good idea to build an asymmetric roof so the water tank is near a wall to help support it. Have a hot- and a cold-water tank on either end of the cabin and put strong roof trusses between them so the roof does not sag in the middle.

By the incenter theorem, the angle bisectors intersect at the center of the incircle. Drop a perpendicular to the ceiling joist and extend it to the other side of the circle so it is a diameter. Measure it with a ruler to determine how large of a tank you can have. Orange belts will learn of the right triangle incircle theorem, that the **indiameter** is the sum of the legs minus the hypotenuse. Because our roof's apex angle is right, this equation would give the same answer.

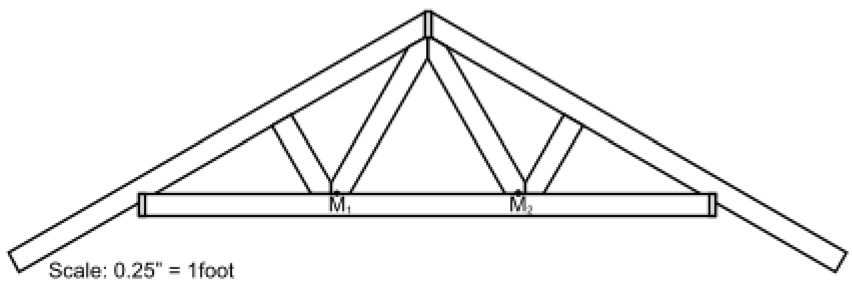
If you are off the grid because of a war, dig into the reverse slope (away from the enemy) and build your cabin with the steeply pitched roof facing uphill with gun ports so you can fire on the enemy should they crest the hill. Use the strong roof truss every $16''$ throughout and with double-thick plywood. Pile dirt and rocks onto the roof to protect it from mortar shells. If a tank crests the hill, run. They cannot depress their gun to shoot downhill, but they can drive over you.

Orange Belt Geometry for Construction Workers, next, originally came at the end of the orange-belt chapter, which is logical, because the approximations to the Tudor arch do not exist in **non-Euclidean geometry**. However, few construction workers survive orange belt, so I moved this section forward. Purists can restrict themselves to the classic arches and bridges, while those who must approximate can pretend that they are orange belts and have the parallel postulate. The Tudor builders were working from scratch, but arches today are almost always a façade on an existing structure, so approximations are essential to the modern construction company. Right rectangles do not exist in hyperbolic geometry, so only orange belts can give width to boards. This section will draw plans for Fink and asymmetrical trusses using $2'' \times 6''$ boards.

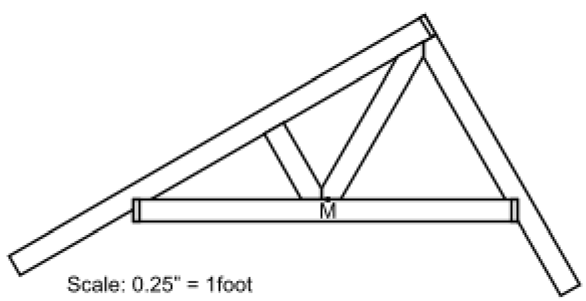
Orange Belt Geometry for Construction Workers

The single best piece of advice that I can offer construction workers is to use the metric system.[52] (I have seen plumbers almost come to blows because they messed up the laying of sewer pipe at $1''$ drop every $8'$. A drop of one centimeter every meter – a 1% downgrade – is the same thing, but it is a *lot* easier.) I would have scaled these roof truss figures at 1 cm = 1 foot, but they would not fit in the book. Construction 3.3 draws a line parallel to a given line through a point not on the line, but you can do this now with a rolling ruler or by sliding a plastic triangle along a straight-edge. I encourage you to draw these roof trusses at 1 cm = 1 foot now, before next semester.

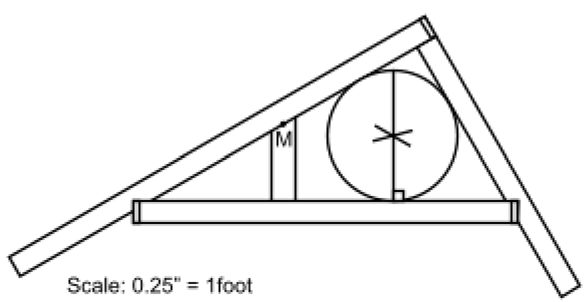
Fink Roof Truss



Asymmetrical Roof Truss, Strong



Asymmetrical Roof Truss, Hollow



[52] Use u, v, w for the legs and hypotenuse of a right triangle; a, b, c are for $ax^2 + bx + c = 0$ and $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Suppose that, while a house is being built, the workers must cross a ditch that is bridged with a 2” X 12” X 12’ board. It is bouncy underfoot and dangerously close to breaking when a fat man with a wheelbarrow full of sand is at its center. Inscribe an arc in the board from one corner to the center point of the other side and across to the other corner. Cut along the arc, steam bend another board over it and attach the two boards with corner braces. The radius of an arc inscribed in a 2” X 12” X 12’ board is 19’ 3”[53] because, after being planed, it is 1.5” X 11.5” X 12’.

To inscribe an arc in a rectangle, find the center using the circumcenter theorem with your three points being two corners on the same side and the midpoint of the opposite side. For aesthetic reasons, it is important that the several arcs in your design all have the same radius. So, if one arc is given, choose any three widely spaced points on it and apply the circumcenter theorem. This is Euclidean, but only because we are fitting it to an existing rectangle.

Arches with one center like above are called Roman and their chord is called the spring line.[54] A Gothic arch has two centers; if they are at the endpoints of the spring line, it is called equilateral. If the centers are on the extensions of the spring line, it is called lancet; if inside, it is called deep.

If you know the height and width, then draw mediators of the legs of the triangle defined by the end points of the spring line and the height of the arc above its midpoint. Where the mediators intersect the spring line or its extension are the two arc centers. If they intersect the spring line before they intersect each other, the centers must be below the spring line equally distant past where the mediators intersect each other. How far past is indeterminate. This is called a pointed Roman arch because it does not look very Gothic. The Goths were a long time ago and use of their arches gives a building a medieval look. The ogee arch is Gothic. It is very pointed, which symbolizes prayers going up to the Lord in the sky.[55] The onion dome is like a bulging ogee arch.

Ogee Arch. Circumscribe a rectangle around an equilateral triangle. Draw arcs centered at the midpoint of the base and the endpoints of the side opposite the base with a radius of half the base. The arch is the former arc from the base endpoints to the leg midpoints, and the latter arc from the leg midpoints to the triangle apex. These arcs and the base enclose a window. ■

A Roman arch is often seen above a window, door, or gate. It is squat, which is fine if it is on top of a rectangle; but, if it is the entire opening, its corners are too sharp. A Tudor arch is also squat, but it has upright corners and works well for wide entrances to big buildings. The classic Tudor arch is neutral geometry, but the approximations, C. 2.5, and the generic arch, are Euclidean.

---

[53] An arc inscribed in a $w \times h$ rectangle has radius $r = \frac{4h^2+w^2}{8h}$ because, by Pythagoras, $(r-h)^2 + \left(\frac{w}{2}\right)^2 = r^2$.

[54] I do not know the origin of this term and, yes, it should be spring segment, but we will stick with tradition.

[55] Can a geometer construct an ogee arch using only neutral geometry? Under what conditions is this possible?

Tudor Arch. Quadrisect the spring line, $\overline{EF}$. Construct an equilateral triangle $\overline{M_{EM_{EF}}JM_{M_{EF}F}}$ with J below the spring line. Extend $\overrightarrow{M_{EM_{EF}}J}$ to K so $\overline{JK} = \overline{M_{EM_{EF}}J}$ and $\overrightarrow{M_{M_{EF}F}J}$ to L so $\overline{JL} = \overline{M_{M_{EF}F}J}$. Draw arcs (called haunch arcs) centered at $M_{EM_{EF}}$ and $M_{M_{EF}F}$ with radii $\overline{M_{EM_{EF}}E}$ and $\overline{M_{M_{EF}F}F}$ to intersect $\overrightarrow{JM_{EM_{EF}}}$ at M and $\overrightarrow{JM_{M_{EF}F}}$ at N, respectively. Draw arcs (called crown arcs) centered at K and L with radii $\overline{KM}$ and $\overline{LN}$ to intersect at the apex. ■

Suppose that a beam bridge made of reinforced concrete spans a canal four meters wide and it is h meters above the concrete sides of the canal. This is ugly, so the city has hired you, a mason, to construct a façade to make it appear that the bridge is a Tudor arch made entirely of brick.

Like the expedient bridge over a ditch (above), we are fitting to an existing rectangle, so this is Euclidean. There are many arches with four centers but, if we are going to call ours Tudor, then it must coincide with the classic one when given a height $\frac{\sqrt{6}-\sqrt{3}}{2}$ of the width. Also, we must quadrisect the spring line regardless of the height. If the centers of the haunch arcs are near the edges of the spring line, the arch would look Gothic but with rounded corners. If the centers of the haunch arcs are near the midpoint of the spring line, the arch would look like a pointed Roman. But putting the centers of the haunch arcs on the quartile points makes it look Tudor.

Construction 2.5 *Construct a Tudor arch given a height and width approximately that of the classic Tudor arch.*

> ***Solution***
>
> Quadrisect the spring line, $\overline{EF}$. Raise a perpendicular to $\overleftrightarrow{EF}$ from M_{EF} and lay off $\overline{M_{EF}K}$ to be the given height. From F, raise a perpendicular to $\overleftrightarrow{EF}$ on the same side as K and lay off $\overline{FJ} = \overline{FM_{M_{EF}F}}$. Connect $\overline{KJ}$. From K, raise a perpendicular to $\overleftrightarrow{KJ}$ on the same side as M_{EF} and lay off $\overline{KL} = \overline{FM_{M_{EF}F}}$. Connect $\overline{M_{M_{EF}F}L}$. Where its mediator intersects $\overrightarrow{KL}$ is the center of the crown arc, O, and its radius is $\overline{OK}$. The haunch arc center is $M_{M_{EF}F}$ and has radius $\overline{FM_{M_{EF}F}}$; the arcs meet on $\overrightarrow{OM_{M_{EF}F}}$. Analogously on the other side. ■

Proof that this coincides with the Tudor arch when given a height $\frac{\sqrt{6}-\sqrt{3}}{2}$ of the width comes later.

Suppose that the Tudor arch we just designed was not a façade but was meant to support the roadbed. Would it work? No. Brick is not as strong as stone and – far more fatal to the design – the mortar between them is much weaker than the bricks. Since all the bricks are rectangular, they can only describe an arc if there is mortar between them and it is in a wedge shape. Brick arches look elegant and are recommended for aesthetic reasons, but they are not weight bearing.

The Romans cut stones into isosceles triangle frustums (isosceles triangles cut by lines parallel to their bases) for bridge construction. This works in areas free of earthquakes, but, because gravity presses them together, if the bridge is jolted, it all comes crashing down. In modern times, bolting the stones together would make it earthquake resistant, but a more fundamental flaw is that it is only strong if it is most of a semicircle, which is too steep for vehicles. Flatter Roman arches are weak in the center, and they push outwards as well as down, so they must have sturdy foundations. Also, if the river rises, flood water pushes on the sides of the bridge near the banks and, if the river is flowing fast, it can push the stones out of place and cause a collapse.

Engineers wish for an arch flat enough for vehicles to climb over but with upright corners, so the weight presses straight down, and the roadbed is above high waters. An ellipse would work, but this is not feasible because every stone would be unique, and it is not practical to readjust the saw for each cut. However, if you have two saws, they can be calibrated to cut the triangle frustums needed for arcs of two different radii. Thus, the Tudors approximated an ellipse with haunch arcs of one radius and a single crown arc that goes all the way across rather than meeting in a pointed apex. With modern construction, Tudor bridges are strong enough for truck traffic.

Tudor Bridge. Quadrisect the spring line, $\overline{EF}$. Construct an equilateral triangle $\overline{M_{EM_{EF}}JM_{M_{EF}F}}$ with J below the spring line. Draw arcs (called haunch arcs) centered at $M_{EM_{EF}}$ and $M_{M_{EF}F}$ with radii $\overline{M_{EM_{EF}}E}$ and $\overline{M_{M_{EF}F}F}$ to intersect $\overrightarrow{JM_{EM_{EF}}}$ at M and $\overrightarrow{JM_{M_{EF}F}}$ at N, respectively. Draw an arc (called the crown arc) centered at J with radius $\overline{JM} = \overline{JN}$. ■

The Tudor bridge is just like the Tudor arch except, instead of extending the legs of the equilateral triangle an equal distance to find the two centers of the crown arcs, we take the apex of the equilateral triangle to be the center of the single crown arc. This is neutral geometry. An isosceles triangle would make the bridge flatter or steeper, but this is not recommended.

Is this design useful to carpenters? No. Archimedes, a Syracuse mathematician who studied in Alexandria shortly after the time of Euclid, invented what is now called the Archimedes' trammel. It draws ellipses or, if a router is attached to the arm, cuts them. Proof that this works is beyond the scope of this book, but it does, so carpenters have no need for approximating an ellipse with arcs. Note that the word trammel, without the adjective, refers to a board with two awls clamped to it. Pushing one into wood and rotating the board allows the other to scratch an arc, so the device functions just like a geometer's compass, but it can reach across a sheet of plywood.

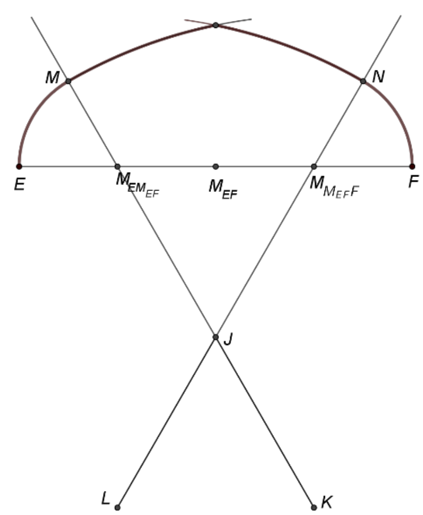
Geometers drawing poster-size figures can buy a pencil trammel at a shop for wood workers. It is time consuming to adjust, but it draws circles larger than the 54 cm of the Alvin 702V.

Construction 2.5 requires that the height and width be approximately that of the classic Tudor arch; that is, in the range $\frac{1}{4}w < h < \frac{\sqrt{2}}{2}w$. To draw an arch to an arbitrary height requires that the radius of the haunch arc be a function of height; specifically, $\frac{2}{3}h$. This is called generic because squat/tall generic arches look like Roman/Gothic arches with rounded corners and, for the same height as the classic Tudor arch, the generic arch looks Tudorish, but it is not exactly the same.

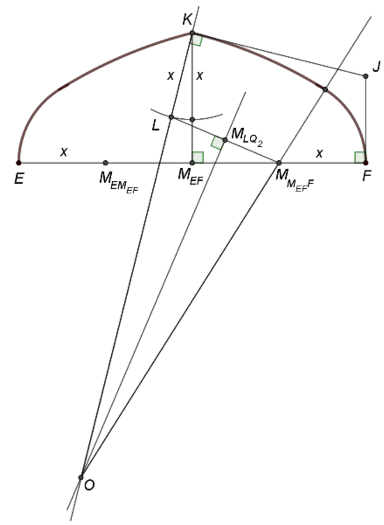
Two thirds of $\frac{\sqrt{6}-\sqrt{3}}{2}$ is $\frac{\sqrt{6}-\sqrt{3}}{3} \approx 23.91\% \neq 25\%$ of the width, so it is not mathematically Tudor.

We will cite C. 3.11; construction workers must skip ahead, or just take it as a cook-book recipe.

Generic Arch. Bisect the spring line, $\overline{EF}$, and raise a perpendicular from M_{EF} to K so that $\overline{M_{EF}K}$ is the given height. By C. 3.11, trisect $\overline{M_{EF}K}$ and call two-thirds the height x. Locate $M_{EM_{EF}}$ and $M_{M_{EF}F}$ on the spring line so $\overline{EM_{EM_{EF}}}$ and $\overline{M_{M_{EF}F}F}$ are x long. From F, raise a perpendicular to $\overleftrightarrow{EF}$ on the same side as K and lay off $\overline{FJ} = \overline{FM_{M_{EF}F}}$. Connect $\overline{KJ}$. From K, raise a perpendicular to $\overleftrightarrow{KJ}$ on the same side as M_{EF} and lay off $\overline{KL} = \overline{FM_{M_{EF}F}}$. Connect $\overline{M_{M_{EF}F}L}$. Where its mediator intersects $\overrightarrow{KL}$ is the center of the crown arc, O, and its radius is $\overline{OK}$. The haunch arc is centered at $M_{M_{EF}F}$ and has radius $\overline{FM_{M_{EF}F}}$; the arcs meet on $\overrightarrow{OM_{M_{EF}F}}$. Analogously on the other side. ■



Tudor Arch



Generic Arch

Problem 2.44 *Prove that, for any $x < h$, the generic arch's haunch and crown arcs are tangent.*

Problem 2.45 *A sewer pipe at a 1% downgrade is 1 m above the city line, which is 5 m away. You will use two $22.5°$ elbows and then enter the city line at a 1% downgrade. If pipe is cut 3 cm from the bend in the elbow, how long is the hypotenuse pipe? Then, how far to the city line?*

On the Importance of Not Neglecting the Third Dimension

When I was in high school (1984) I entered the *Balsa Bridge Building Contest*[56] with a truss design. The roadbed twisted and the trusses turned into an S-shape when viewed from the top; it did not break, but it very quickly deformed enough to fail. The side view showed triangles galore, but the downward view showed no triangles, just some cross braces between the trusses. My sad balsa wood bridge should serve as a cautionary tale about seeing things in only two dimensions.

How many students were rolling their eyes when they got to problem 2.14 about Poe's pendulum? You should have been. Generations of students have read *The Pit and the Pendulum* in their American Literature class with nary a dissenting voice. When restricted to the plane, our solution to problem 2.14 is sound; but is that big blade really restricted to the plane? Monkish ingenuity notwithstanding, when the blade is going fast near the nadir of its swing, if it is out of line with its motion, the air will push the leading edge even more out of line. The trailing edge does not act as a fin because air that hits the leading edge slides down the length of the blade and prevents still air from pushing the trailing edge back into line. That big blade will start spinning on its cable and will eventually be doing a languid circle around the man strapped to the floor, who is probably by then loudly mocking his executioners for their weak engineering skills.

Another unquestioningly accepted theory is IS-LM Analysis. This is two equations, the income identity, and the demand for real money holdings, in two variables, national income and the interest rate. When students point out that the obvious policy prescription is to print money to hold the interest rate at zero, economists talk about how these curves "shift" in response to inflation. Bad move! Curves do not shift. What is actually happening is that there are three equations in three unknowns and economists initially tried to assume away the third dimension by declaring the price level to be constant. When that did not work, they turned their problem into a sequence of problems, each with a constant price level but, in some hazily defined way, shifted from the previous problem. Their near-religious devotion to "data" blinds them to the fact that IS-LM Analysis is based on deductive logic, not statistics, and that it is missing an axiom.

My *Axiomatic Theory of Economics*[57] has its own axiom set, distinct from the two-going-on-three axioms of John Hicks' IS-LM Analysis. But the point is, if your problem is three dimensional, then model it with three equations and then simultaneously solve them for all three variables. If this seems daunting, then learn the needed math. Looking at a two-dimensional slice of a three-dimensional problem because the graphs are easy to draw and/or because you only know how to solve two equations in two unknowns does not result in a simplification; it results in nonsense.

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[56] In my day, we could use sheets of balsa wood; now they disallow this: www.balsabridge.com

[57] www.researchgate.net/profile/Victor_Aguilar4/publication/270687116_AXIOMATIC_THEORY_of_ECONOMICS

Advanced Yellow Belt Geometry: Quadrilaterals

In my opinion, students take too many vacations. There are 52 weeks in the year no matter how you slice it, so every week off in the middle of the school year is a week less of summer vacation. This is the principal reason why schools struggle to retain farm boys; school starts in the middle of the fall harvest, so the children cannot be there on day one and, already a week behind when they do show up, they flunk out. For city kids, it just exchanges a summer week at the skate park for a winter week at the rec center basketball court. It does not help them learn. In a week's time, they have forgotten everything they learned before vacation, so their teachers waste yet another week reviewing material that the students once had thoroughly in mind.

Thus, I refer to fall break as "yellow belt preparation week," assign homework and keep office hours to help the students with it. It is also why I refer to early release days as "double homework days;" just because the teachers will spend the afternoon eating doughnuts and listening to sanctimonious speeches is no reason for the students not to have their noses to the grindstone!

Over Christmas break, you might consider reading the beginning of the orange-belt chapter; you will learn a lot more about your friend, the incircle, and you will get to meet his out-of-town cousins, the excircles. The parallel postulate will be introduced to prove the transversal and angle sum theorems, and students will prove that a **Lambert quadrilateral**, which has three right angles, is a right rectangle. But here in the yellow-belt chapter, before the parallel postulate has been introduced, this is not necessarily true; in hyperbolic geometry, the fourth angle is acute.

A Saccheri quadrilateral has two opposite sides equal and perpendicular to the base. By part three of Saccheri theorem I (below, from the white-belt exit exam), a Saccheri quadrilateral's base mediator cuts it into two Lambert quadrilaterals, which we will soon prove to be congruent. Knowing this is not needed to understand orange-belt *Geometry–Do*, but it helps tie yellow- and orange-belt study together, and it will be needed when you become a green belt.

Saccheri Theorem I

If $\overline{EFGH}$ is a Saccheri quadrilateral, so $\angle E = \angle F = \rho$ and $\overline{EH} = \overline{FG}$, (1) $\overline{EG} = \overline{FH}$; (2) $\angle G = \angle H$; (3) $\overleftrightarrow{M_{EF}M_{GH}} \perp \overleftrightarrow{EF}$ and $\overleftrightarrow{M_{EF}M_{GH}} \perp \overleftrightarrow{GH}$; (4) The mediators of the base and the summit coincide.

Saccheri was a Jesuit priest whose book, *Euclides ab omne naevo vindi*, was meant to "vindicate Euclid of every blemish," by which he meant that he would prove Euclid's fifth postulate redundant. Success would have actually been a black eye for Euclid – redundant postulates are no virtue – but Saccheri failed. But while he was failing he proved theorems that, over a century later, would serve as the foundation for what Bolyai would call absolute (neutral) geometry.

Suppose that $\overline{EFGH}$ is such that $\angle E = \angle F = \rho$, but this is all that is known about it, so it is only part ways towards being a Saccheri quadrilateral. What can we say about $\overline{EFGH}$?

Two Right Angles Quadrilateral Theorem

Given $\overline{EFGH}$ such that $\angle E = \angle F = \rho$, if $\overline{HE} < \overline{FG}$, then $\angle G < \angle H$.

Proof

$\overline{HE} < \overline{FG}$ implies that there is a point M between F and G such that $\overline{HE} = \overline{FM}$. $\overline{EFMH}$ is Saccheri with base $\overline{EF}$; by part two of Saccheri theorem I, $\angle FMH = \angle EHM$. By the exterior angle inequality theorem, $\angle G < \angle FMH$. $\angle EHM < \angle H$ by the interior angle axiom. Thus, $\angle G < \angle H$, by transitivity. ■

Two Right Angles Quadrilateral Theorem Converse

Given $\overline{EFGH}$ such that $\angle E = \angle F = \rho$, if $\angle G < \angle H$, then $\overline{HE} < \overline{FG}$.

Proof

If $\overline{HE} = \overline{FG}$, then $\overline{EFGH}$ is Saccheri with base $\overline{EF}$ and $\angle G = \angle H$ by part two of Saccheri theorem I, which contradicts $\angle G < \angle H$. If $\overline{FG} < \overline{HE}$, then $\angle H < \angle G$ by the two right angles quadrilateral theorem, which contradicts $\angle G < \angle H$. Thus, $\overline{HE} < \overline{FG}$, by trichotomy. ■

A **Lambert quadrilateral** has three right angles. From part three of Saccheri theorem I, if $\overline{EFGH}$ is a Saccheri quadrilateral, then $\overline{EM_{EF}M_{GH}H}$ and $\overline{FM_{EF}M_{GH}G}$ are Lambert quadrilaterals. But are they congruent? Recall that a quadrilateral is a union of two adjacent triangles such that it is convex; congruence holds if and only if both pairs of definitional triangles are congruent. Since we have proven SASAS, we will cite it, though there are many ways to prove this theorem.

Saccheri and Lambert Theorem

If $\overline{EFGH}$ is a Saccheri quadrilateral with base $\overline{EF}$, then $\overleftrightarrow{M_{EF}M_{GH}}$ cuts it into two congruent Lambert quadrilaterals, $\overline{EM_{EF}M_{GH}H} \cong \overline{FM_{EF}M_{GH}G}$.

Proof

$\overline{EM_{EF}M_{GH}H}$ and $\overline{FM_{EF}M_{GH}G}$ are Lambert quadrilaterals by part three of Saccheri theorem I. By SASAS, $\overline{EM_{EF}M_{GH}H} \cong \overline{FM_{EF}M_{GH}G}$. ■

In the orange-belt chapter, the Lambert theorem will state that Lambert quadrilaterals are right rectangles. But, without the parallel postulate, this is not necessarily true. So, what is true?

Three Right Angles Quadrilateral Theorem

Given $\overline{EFGH}$ such that $\angle E = \angle F = \angle G = \rho$, then

1. *If $\angle H$ is right, then the opposite sides of $\overline{EFGH}$ are equal;*
2. *If $\angle H$ is acute, then each side of $\angle H$ is greater than its opposite side.*

Proof

1. Suppose $\overline{HE} < \overline{FG}$. By the two right angles quadrilateral theorem, $\angle G < \angle H$, which contradicts $\angle G = \angle H$. Analogously, $\overline{FG} < \overline{HE}$ contradicts $\angle G = \angle H$. Thus, $\overline{HE} = \overline{FG}$ by trichotomy. Analogously, $\overline{EF} = \overline{GH}$.

2. $\angle E = \angle F = \rho$ and $\angle H < \angle G$. By the two right angles quadrilateral theorem converse, $\overline{FG} < \overline{HE}$. Analogously, $\overline{EF} < \overline{GH}$. ■

We do not consider what happens if $\angle H$ is obtuse because this never happens; it is right in Euclidean geometry and acute in hyperbolic geometry. It is obtuse in elliptic geometry, but this is studied independently of the others because Euclidean and hyperbolic geometry differ only in their parallel postulate; the former assumes that there is exactly one line through a point parallel to another line, and the latter assumes that there are always more than one. All the other postulates are shared; so, here in neutral geometry, we are assuming all the other postulates, and that there is at least one line through a point parallel to another line, but possibly more.

In elliptic geometry, there are no parallel lines. But this assumption also takes out some of the other postulates, besides just the parallel postulate. In this book, lines are defined to be infinite. The line postulate allows us to extend a segment as far as necessary to locate any point needed in a construction. In elliptic geometry, if you extend a segment far enough, it wraps around the globe and your construction overlaps itself. In practical navigation problems, like plotting an airplane's course from Tokyo to Los Angeles, this is not an issue, but it must be considered to give a solid theoretical foundation to the endeavor. But we will not be doing any elliptic geometry.

The Saccheri-Legendre theorem will prove that the angle sum of a triangle is less than or equal to a straight angle; thus, a quadrilateral's angle sum is less than or equal to two straight angles.

Adrien-Marie Legendre was a French mathematician who is best known today for Legendre polynomials. *Geometry Informs the Numerical Analysis of Error in Computations* discusses Chebyshev polynomials, which are orthogonal in the interval $-1 \leq x \leq 1$, as are Legendre polynomials. Unlike Chebyshev's polynomials, which are used to approximate a variety of unrelated functions, Legendre's polynomials are used to solve problems that arise in electrical engineering. This is beyond high school, but students should at least know the men's names.

Lemma 2.6

Given $\overline{EFG}$, if $\angle M_{FG}EF \leq \angle M_{FG}EG$, then $\angle M_{FG}EF \leq \frac{1}{2}\angle E$.

Recall the notation that $\alpha,\ \beta,\ \gamma,\ \delta$ are usually $\angle E, \angle F, \angle G, \angle H$, which will be the case here.

Saccheri–Legendre Theorem

Interior angles of a triangle sum to one straight angle or less; that is, $\alpha + \beta + \gamma \leq \sigma$.

> ***Proof***
>
> Given $\overline{EFG}$, label the vertices so $\angle M_{FG}EF \leq \angle M_{FG}EG$. Assume $\alpha + \beta + \gamma = \sigma + \varepsilon$; that is, the angle sum is more than straight. Extend $\overrightarrow{EM_{FG}}$ that much again to H, so M_{FG} is the midpoint of $\overline{EH}$. By SAS, $\overline{EM_{FG}G} \cong \overline{HM_{FG}F}$, so $\angle HFG = \gamma$ and $\angle FHE = \angle GEH$, which we will call δ. Let $\delta' = \alpha - \delta$. $\overline{EFH}$ has angle sum $\delta + \delta' + \beta + \gamma = \alpha + \beta + \gamma$, which is the same angle sum that $\overline{EFG}$ has.
>
> Re-label $\overline{EFH}$ as $\overline{E_1F_1G_1}$ so E_1 is the vertex with angle δ' and $\angle M_{F_1G_1}E_1F_1 \leq \angle M_{F_1G_1}E_1G_1$. $\overline{E_1F_1G_1}$ has the same angle sum as $\overline{EFG}$ and, by lemma 2.6, $\angle E_1 \leq \frac{1}{2}\angle E$. Repeat to construct $\overline{E_2F_2G_2}$ that has the same angle sum as $\overline{EFG}$ and $\angle E_2 \leq \frac{1}{4}\angle E$. After n repetitions, $\overline{E_nF_nG_n}$ is constructed to have the same angle sum as $\overline{EFG}$ and $\angle E_n \leq \frac{1}{2^n}\angle E$.
>
> By Archimedes' Axiom, there exists a natural number n such that $\angle E_n \leq \varepsilon$; that is, the repetitive process described above terminates after a finite number of iterations. We will not get stuck in an endless loop. The angle sum of $\overline{E_nF_nG_n}$ is $\sigma + \varepsilon$ and, since $\angle E_n \leq \varepsilon$, $\sigma \leq \angle F_n + \angle G_n$, which contradicts lemma 2.1. Thus, the assumption that the angle sum is more than straight is not true; it is one straight angle or less. ■

In this proof, $\frac{1}{2^n}$ does not require students to know what exponents are or to have the ability to calculate them. It is just a shorthand notation to refer to bisecting an angle n times. Indeed, throughout *Volume One*, $\frac{1}{2}$ just means bisection. It does not mean that we have slyly introduced division and are expecting students to calculate $\frac{p}{q}$ for arbitrary integers p and q.

By the Saccheri-Legendre theorem, a triangle's angle sum is $\sigma - \varepsilon$, where ε is called the **defect**.

Defect Addition Theorem

The defect of a quadrilateral is the sum of the defects of its definitional triangles.

This is also true for adjacent triangles that are assembled so their union is a big triangle. Like all angles, $0 \leq \varepsilon$, so their defects cannot cancel each other out; that is, for the big triangle to have a defect of zero, all the smaller triangles must have defects of zero.

The implication of this is that Euclidean and hyperbolic geometry are mutually exclusive. In the former, all triangles have defects of zero; and, in the latter, all triangles have positive defects. Their exclusivity makes sense because the two sciences are based on different parallel postulates.

Another implication is that Euclidean and hyperbolic geometry are mutually exclusive with regards to the existence of right rectangles. If one right rectangle exists, then all rectangles are right, and we are doing Euclidean geometry. If one rectangle is not right, then none of them are.

If we have declared that there is more than one parallel to a line through a point, then right rectangles do not exist, and we are primarily concerned with Saccheri quadrilaterals and the two congruent Lambert quadrilaterals that they are unions of.

Saccheri Theorem II

If $\overline{EFGH}$ is a Saccheri quadrilateral with base $\overline{EF}$, then

1. $\angle G = \angle H \leq \rho$
2. $\overline{EF} \leq \overline{GH}$
3. $\overline{M_{EF}M_{GH}} \leq \overline{HE}$ *and* $\overline{M_{EF}M_{GH}} \leq \overline{FG}$

Yet another implication is that a big triangle is more defective than any small triangle that can fit inside it. The defect of a triangle is proportional to its area, but proof of this is beyond us because, while Euclidean geometry defines area as the number of right squares that fit in a closed figure, right squares do not exist in hyperbolic geometry. But the implication is that, while Euclidean and hyperbolic geometry are mutually exclusive, if poor visibility limits us to only measuring small triangles and/or our measuring instruments are imprecise, we may not know if geometry is Euclidean or hyperbolic. When excircles are introduced, orange belts will consider a scenario where civil engineers are using them, but they are unsure if geometry is hyperbolic on their scale.

Carl Friedrich Gauss famously measured the angle sum of a triangle with vertices on three mountain peaks. Because the defect of a triangle is proportional to its area, he chose a triangle with as much area as possible so its defect – if it had one – would be measurable with his rather crude 19th century instruments. Because Euclidean and hyperbolic geometry are mutually exclusive, if he had found a defect large enough that it could not be written off as instrument error, then he would have proven conclusively that the world has a hyperbolic geometry. But the constant of proportionality between triangle area and defect, in standard units like hectares and radians, would not be conclusive; it could always be further refined with more accurate instruments. What happened is that he did not find a measurable defect in this big triangle. But this does not prove conclusively that the world has a Euclidean geometry. It is still possible that more accurate instruments and a bigger triangle – perhaps one with vertices on Earth, Jupiter, and Saturn – might conclusively prove that the world has a hyperbolic geometry.

The fact that it has not been and never will be proven conclusively that the world has a Euclidean geometry is why it is important that *Geometry–Do* begins with neutral geometry. The principal reason that we study geometry at all is not just for practical applications like laying ambushes with heavy machine guns – though in certain situations that can be very useful – but to free our minds of the tacit assumptions that run rampant in soft sciences like economics. Geometers are better than that! We take nothing for granted – not even the Euclidean nature of our world.

Now let us consider what is certainly the most common geometric figure used by practical men.

Rectangle Theorem

If $\overline{EFGH}$ is a Saccheri quadrilateral with base $\overline{EF}$, let G_{EF} and H_{EF} be reflections of G and H around $\overleftrightarrow{EF}$ so $\overline{EFGH} \cong \overline{EFG_{EF}H_{EF}}$. Then the following holds true:

1. *$\overline{HGG_{EF}H_{EF}}$ is a rectangle.*
2. *Both **bimedians** of $\overline{HGG_{EF}H_{EF}}$ cut it into two congruent Saccheri quadrilaterals.*
3. *Opposite sides of $\overline{HGG_{EF}H_{EF}}$ are equal.*
4. *Bimedians of $\overline{HGG_{EF}H_{EF}}$ are mediators of each other.*
5. *Diagonals of $\overline{HGG_{EF}H_{EF}}$ are equal and bisect each other.*
6. *Perpendiculars dropped on diagonals from the vertices of $\overline{HGG_{EF}H_{EF}}$ are equal.*
7. *Bimedians of $\overline{HGG_{EF}H_{EF}}$ are less than or equal to the sides they do not cut.*

Proof

1. By part two of Saccheri theorem I, $\angle G = \angle H$, and, by congruence, $\angle G_{EF} = \angle H_{EF}$ are the same, so all four angles are equal.

2. By part three of Saccheri theorem I, both bimedians are perpendicular to their sides, so they both cut the rectangle into two congruent Saccheri quadrilaterals.

3. Opposite sides are equal by construction and congruency.

4. Bimedians are mediators of each other by the Saccheri and Lambert theorem.

5. Diagonals are equal by SAS. By SAS, $\overline{TEH_{EF}} \cong \overline{TEH} \cong \overline{TFG} \cong \overline{TFG_{EF}}$ with T the bi-medial point. Thus, $\overline{TH_{EF}} = \overline{TH} = \overline{TG} = \overline{TG_{EF}}$; the diagonals bisect each other.

6. If $\overline{GH} = \overline{GG_{EF}}$, then, by part five. If $\overline{GH} < \overline{GG_{EF}}$, then, by the above congruencies, $\angle EH_{EF}T = \angle EHT = \angle FGT = \angle FG_{EF}T$. Let H', G', G'_{EF}, H'_{EF} be the feet of perpendiculars dropped on the diagonals from the vertices of $\overline{HGG_{EF}H_{EF}}$. By AAS, $\overline{G'G_{EF}G} \cong \overline{H'H_{EF}H} \cong \overline{G'_{EF}GG_{EF}} \cong \overline{H'_{EF}HH_{EF}}$. $\overline{G'G} = \overline{H'H} = \overline{G'_{EF}G_{EF}} = \overline{H'_{EF}H_{EF}}$. If $\overline{GG_{EF}} < \overline{GH}$, then analogously with the other four right triangles being congruent.

7. This is implied by the three right angles quadrilateral theorem applied to each of the four Lambert quadrilaterals, and then the lengths doubled. ■

In Euclidean geometry, part seven would say "equal," but otherwise rectangles in hyperbolic geometry differ from right rectangles only in their vertex angles being acute. The big difference, of course, is that Lobachevskians cannot fit a bunch of them together in a grid as Descartes did.

The astute student will have noticed that part one has a converse that a conscientious textbook author – *Ahem!* – would have proven. The converse is indeed provably true, but only for rectangles that exist. In Euclidean geometry, all rectangles exist; you can set the height and the width to be anything you want. A kilometer wide and a centimeter high if it pleases you! But, in hyperbolic geometry, once you have chosen an angle, you are restricted to choosing only those heights and widths such that the sum of the areas of the definitional triangles is in the correct proportion to the sum of their defects. By "correct proportion," I mean the one determined by an experiment conducted with the best instruments on mountain peaks hundreds of klicks apart. Calculating areas in a hyperbolic world and carrying out such an experiment is beyond us.

Rectangles are important, but the most often cited orange-belt theorem is the mid-segment theorem. There is an analogous yellow-belt theorem, which uses $\leq$ instead of $=$.

Mid–Segment Theorem (Neutral Geometry)

1. *The mid-segment connecting the legs of a triangle is less than or equal to half the base.*
2. *The extension of the mid-segment does not intersect the extension of the base.*

Proof

1. Construct the same figure as in the mid-segment and mediator theorem; $\overline{E''F''FE}$ is Saccheri with base $\overline{E''F''}$. By AAS, $\overline{E''M_{GE}E} \cong \overline{G''M_{GE}G}$ and $\overline{F''M_{FG}F} \cong \overline{G''M_{FG}G}$; thus, $\overline{E''M_{GE}} = \overline{G''M_{GE}}$ and $\overline{F''M_{FG}} = \overline{G''M_{FG}}$. The result by Saccheri theorem II #2.

2. Suppose $P := \overleftrightarrow{EF} \cap \overleftrightarrow{E''F''}$ exists with E between P and F. By Saccheri theorem II #1, $\angle E \leq \rho$, so its supplement is right or obtuse. But $\angle E'' = \rho$, so the angle sum of $\overline{E''EP}$ is greater than a straight angle, which contradicts the Saccheri-Legendre theorem. ■

The mid-segment theorem is proven just a few pages into orange belt, so proving the neutral geometry version now helps tie yellow- and orange-belt together. There are also analogous neutral geometry versions of the best-known green-belt theorems. They are not hard to prove, but they will be left as exercises for green belts to return to when their context becomes clearer.

Thales' Diameter Theorem (Neutral Geometry)
A diameter subtends an angle less than or equal to a right angle.

Inscribed Angle Theorem (Neutral Geometry)
Two chords that share an endpoint make an angle less than or equal to half the central angle of their arc.

Cyclic Quadrilateral Theorem (Neutral Geometry)
If a quadrilateral is cyclic, then the sums of its opposite angles are equal.

Proving the following conditions for Lambert quadrilaterals, $\angle E = \angle F = \angle G = \rho$, or Saccheri quadrilaterals, $\angle E = \angle F = \rho$ and $\overline{HE} = \overline{FG}$, to be congruent makes good homework problems.

Lambert quadrilaterals $\overline{EFGH} \cong \overline{E''F''G''H''}$		Saccheri quadrilaterals $\overline{EFGH} \cong \overline{E''F''G''H''}$	
$\overline{EF} = \overline{E''F''}$ and $\overline{FG} = \overline{F''G''}$	$\overline{EH} = \overline{E''H''}$ and $\overline{FG} = \overline{F''G''}$	$\overline{EF} = \overline{E''F''}$ and $\overline{FG} = \overline{F''G''}$	$\overline{EF} = \overline{E''F''}$ and $\angle G = \angle G''$
$\overline{EF} = \overline{E''F''}$ and $\overline{EH} = \overline{E''H''}$	$\overline{EH} = \overline{E''H''}$ and $\overline{GH} = \overline{G''H''}$	$\overline{EF} = \overline{E''F''}$ and $\overline{GH} = \overline{G''H''}$	$\overline{FG} = \overline{F''G''}$ and $\angle G = \angle G''$
$\overline{EF} = \overline{E''F''}$ and $\angle H = \angle H''$	$\overline{EH} = \overline{E''H''}$ and $\angle H = \angle H''$	$\overline{FG} = \overline{F''G''}$ and $\overline{GH} = \overline{G''H''}$	$\overline{GH} = \overline{G''H''}$ and $\angle G = \angle G''$

Green belts will be tasked with helping teach the white- and yellow-belt students; they can assign these congruence proofs. The following two problems are for advanced yellow-belt students.

Problem 2.46
Given $\overline{EFG}$ with $\angle EFG = \rho$, let M'_{EG} be the foot of a perpendicular dropped on $\overline{FG}$ from M_{EG}. Prove that $\overline{M'_{EG}F} \leq \overline{M'_{EG}G}$ and $\overline{M_{EG}F} \leq \frac{1}{2}\overline{EG}$.

Problem 2.47
Given $\angle EPE_1 \neq \sigma$ and P, E, F, G in that order on one ray and P, E_1, F_1, G_1 in that order on the other ray and $\overline{EF} = \overline{E_1F_1}$ and $\overline{FG} = \overline{F_1G_1}$, prove that $M_{EE_1}, M_{FF_1}, M_{GG_1}$ are collinear.

And that is all the advanced yellow-belt geometry that I can think of!

I hope everybody has a merry Christmas and I will see you next year. Maybe Santa Claus will bring you a nice compass to replace the cheap one that you bought at the grocery store. 😉

Geometry Don't (Satire)

Dismayed that his friends are now orange belts and he is still a yellow belt, Johnny Geometer has decided that he will write his own textbook! And he already has a publisher! Based only on the following theorems, *McGoober-Hill Education* – famous for teaching us how to add lengths and angles together – has agreed to give Johnny a \$100,000 cash advance. They are planning to print a million copies of *Geometry Don't*, which will be distributed to every *Common Core* compliant high school in America. Let's have a look at the theorems that so impressed *McGoober-Hill!*

Parallel Theorem
There is a unique line parallel to a given line through a given point.

> ***Proof***
> Given $\overleftrightarrow{EF}$ and a point G not on it, by C. 1.4, drop a perpendicular from G onto $\overleftrightarrow{EF}$ with foot G'. By C. 1.3, raise a perpendicular to $\overleftrightarrow{GG'}$ from G. By the perpendicular length theorem, both constructions of perpendicular lines are unique. Thus, the latter perpendicular is the unique parallel to the given line that goes through the given point. 😊

Transversal Theorem
If a line transverses two parallel lines, the interior angles on one side of the transversal sum to σ.

> ***Proof***
> It is a trichotomy: They either sum to more than σ, less than σ, or exactly σ. If it is more, then the two pairs of supplementary angles sum to more than 2σ, which is impossible. If it is less, then they sum to less than 2σ, which is also impossible. Thus, the result. 😊

Angle Sum Theorem
Interior angles of a triangle sum to one straight angle.

> ***Proof***
> Given $\overline{EFG}$, let P be any point inside $\overline{EF}$ and let x be the angle sum of triangles, which we intend to prove is σ. $\angle E + \angle EGP + \angle GPE = x$ and $\angle F + \angle FGP + \angle GPF = x$ for the two interior triangles. Adding, $(\angle E + \angle EGP + \angle FGP + \angle F) + (\angle GPE + \angle GPF) = 2x$. In the first parenthesis is x because it is a triangle, and in the second parenthesis is σ because the angles are supplementary. Thus, $x + \sigma = 2x$; simplifying, $x = \sigma$. 😊

But, while Bill Gates has the big money, there are people pushing back. You, young geometer, have been hired to review *Geometry Don't*. Do you see any flaws in Johnny's reasoning?

Orange Belt Instruction: Parallelograms

Most geometry problems, both in real life and on exams, are constructions. You are told what characteristics a figure must have to qualify as a correct answer and then you construct that figure, citing theorems along the way to justify each step. For instance, to replicate an angle you construct congruent triangles with the given angle equal in both. This is easy because there is only one characteristic required of the figure. But when there are two, the general technique is to disregard one of them and construct a figure that satisfies just the other; that is, the complete solution is under defined. For instance, if one **condition** is that a segment must have a given endpoint and a given length, then any of the radii of a circle centered at the given point and with the given radius satisfy the condition. This circle is the locus of possible locations for the other endpoint. If this procedure is carried out for each of the two conditions, then the intersection(s) of the loci is/are the solution(s) to the problem; that is, the intersection fully defines the solution.

Parallels and Circle Theorem
Parallel lines that intersect a circle cut off equal chords between the two lines.

This is diameter and chord theorem corollary #1 extended with transitivity, which the parallel postulate now allows us to cite. Also, the parallel postulate allows us to now speak of "the intersection" of two non-parallel lines; $P := \overleftrightarrow{EG} \cap \overleftrightarrow{FH}$, given $\overleftrightarrow{EG} \nparallel \overleftrightarrow{FH}$. It assures existence of P.

Circumcenter Theorem
The mediators of a triangle's sides are concurrent at a point equidistant from the vertices.

> ***Proof***
> By the mediator theorem, any point on the mediator of a side is equidistant from the vertices, including the point where it intersects with the mediator of another side. This point, being on the mediator of the other side, is equidistant from the vertices on either end of it. By transitivity, the other mediator is **concurrent**. This is the circumcenter. ■

For reasons explained more fully in *Orange Belt Geometry for Construction Workers, Revisited*, the mediators of triangle sides are only guaranteed to be concurrent in Euclidean geometry.

Circumcenter Theorem Corollary
Any three noncollinear points fully define a circle.

Suppose that some points are proven to be **concyclic**, and that some other points are also proven to be concyclic; if the two lists of points share three points, then they are all on the same circle.

Construction 3.1 *Locate the center of a circle.* ***(Euclid, Book IV, Prop. 5)***

Solution

Choose any three points on the circle and find the mediator of the segments between any two pairs. Their intersection is the center of the circle. ■

The error in locating the intersection of two lines is least if they are perpendicular. The more widely spaced the three points that define a circle, the less error there is in locating its center. If they make a little triangle – sailors call this a cocked hat – the best guess is the triangle's incenter.

Excenter Theorem

The bisectors of a triangle's interior angle and the angles exterior to the other two angles are concurrent at a point we will call the ***excenter****.*

Proof

By the angle bisector theorem, any point on the angle bisector of a vertex is equidistant from the sides, including the point where it intersects the angle bisector of the exterior angle of another vertex. By transitivity, all three angle bisectors are concurrent. ■

Given $\overline{EFG}$, the incircle is ω_I and the incenter is I. The **excircle** on the bisector of $\angle G$ is ω_X and its excenter is X; analogously, ω_Y and ω_Z have excenters Y and Z, respectively. The feet of the perpendiculars dropped on the triangle sides from the incenter are the incircle's touching points I_E, I_F, I_G. The excircles also touch the triangle at touching points; we will call these Z_E, Y_F, X_G. ω_X touches $\overrightarrow{GE}$ at X_E, and analogously for the others. Most problems involve only one excircle, so they are drawn with the apex as G and the excenter as X. R is the circumradius; r is the **inradius**; r_X, r_Y, r_Z are the **exradii** of $\omega_X, \omega_Y, \omega_Z$; s is the semiperimeter. L_E, L_F, L_G are where the bisector of an interior angle intersects the other side of the circumcircle (centered at O); these points are called long centers, though they will not be formally defined until the red-belt chapter.

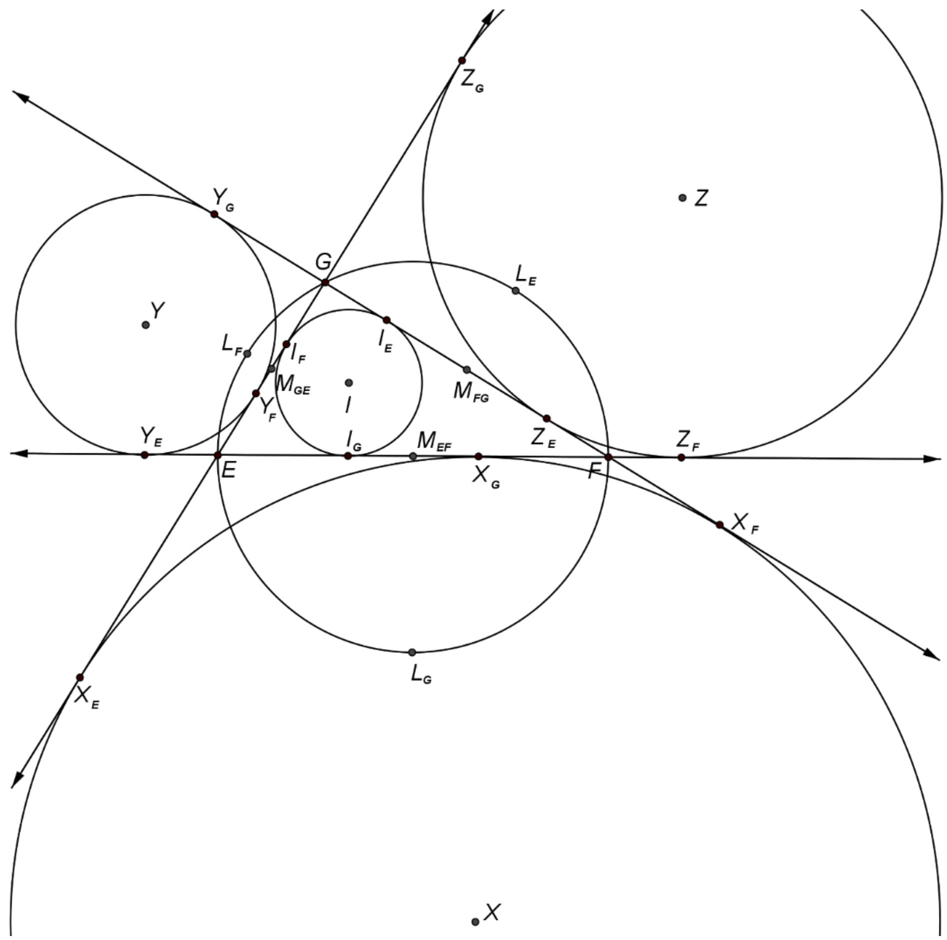
Excircle Theorem

Given $\overline{EFG}$*, the semiperimeter is the distance from* G *to either* X_E *or* X_F*.*

Proof

$\overline{EF} + \overline{FG} + \overline{GE} = \overline{EX_G} + \overline{FX_G} + \overline{FG} + \overline{GE}$	Addition
$= \overline{EX_E} + \overline{FX_F} + \overline{FG} + \overline{GE}$	Two tangents theorem
$= \overline{GX_E} + \overline{GX_F}$	Addition and commutativity

$\overline{GX_E} = \overline{GX_F}$ by the two tangents theorem; thus, the semiperimeter equals either. ■



Incircle and Excircle Theorem

The incircle and excircle touch a triangle side equidistant from its opposite endpoints.

Proof

Given $\overline{EFG}$, construct ω_I and ω_X.

$2\overline{EI_G}$	$= \overline{EI_G} + \overline{EI_F}$	Two tangents theorem
	$= (\overline{EF} - \overline{FI_G}) + (\overline{EG} - \overline{GI_F})$	Subtraction
	$= \overline{EF} + \overline{EG} - \overline{FI_E} - \overline{GI_E}$	Two tangents theorem
	$= \overline{EF} + \overline{EG} - \overline{FG}$	Addition
	$= 2(s - \overline{FG})$	Addition
	$= 2(\overline{X_F G} - \overline{FG})$	Excircle theorem
	$= 2\overline{FX_G}$	Subtraction and two tangents th.

Thus, $\overline{EI_G} = \overline{FX_G}$, and, analogously, $\overline{FI_G} = \overline{EX_G}$. ■

Incircle and Excircle Theorem Corollary

M_{EF} is the midpoint of $\overline{I_G X_G}$.

An **external tangent** is a line tangent to two circles that does not go between their centers. There are two external tangents, and these intersect at the apex of the triangle that the two circles are incircle and excircle to. The term external tangent may also refer, not to the whole line, but just to the segment of it between the points where it touches the two circles. The context will make this clear; if we refer to length, we mean the segment. An **internal tangent** is a line tangent to two disjoint circles that goes between their centers, or the segment between its two touching points. There are two external tangents and one or two internal tangents; one if the circles touch. A **cut tangent** is the segment of an internal tangent that is cut out by the two external tangents.

Constructing external and internal tangents to two disjoint circles of different radii is easy, but it waits for more orange-belt theorems. Here, the circles and their tangents are just given to us.

External Tangents Theorem

The two external tangents to two circles are equal in length.

> ***Proof***
>
> Given $\overline{EFG}$ and external tangents of ω_I and ω_X, by the two tangents theorem, $\overline{GI_F} = \overline{GI_E}$ and $\overline{GX_E} = \overline{GX_F}$. Thus, $\overline{I_FX_E} = \overline{I_EX_F}$, by subtraction. ■

Cut Tangents Theorem

Cut tangents equal external tangents.

> ***Proof***
>
> Given $\overline{EFG}$, then $\overline{EF}$ is the internal tangent of ω_I and ω_X. I_G and X_G are on it.
>
> | | |
> |---|---|
> | $\overline{EF} = \overline{EI_G} + \overline{FI_G}$ | Addition |
> | $= \overline{EI_G} + \overline{EX_G}$ | Incircle and excircle theorem |
> | $= \overline{EI_F} + \overline{EX_E}$ | Two tangents theorem |
> | $= \overline{I_FX_E}$ | Addition |
>
> This works for E, I_G, X_G, F in this order, this order, E, X_G, I_G, F, or if $I_G \equiv X_G$. ■

Excircle Theorem Corollaries

1. $r_X + r_Y + r_Z = r + 4R$ — *The three exradii are the inradius and four circumradii.*
2. $\overline{GI_E} = \overline{GI_F} = s - \overline{EF}$ — *The distance from G to the touching points of ω_I.*
3. $\overline{I_GX_G} = \left|\overline{FG} - \overline{EG}\right|$ — *The distance between where ω_I and ω_X touch $\overline{EF}$.*
4. $\overline{I_FX_E} = \overline{I_EX_F} = \overline{EF}$ — *The distance between where ω_I and ω_X touch $\overleftrightarrow{EG}$ or $\overleftrightarrow{FG}$.*
5. $\overline{Y_EZ_F} = \overline{EG} + \overline{FG}$ — *The distance between where ω_Y and ω_Z touch $\overleftrightarrow{EF}$.*
6. $\overline{Y_FZ_G} = \overline{Z_EY_G} = \overline{EF}$ — *The distance between where ω_Y and ω_Z touch $\overleftrightarrow{EG}$ or $\overleftrightarrow{FG}$.*

Proofs are left as exercises. All of them are straightforward applications of the four preceding theorems. None of these six statements are intuitive. This is why *Geometry–Do* students appreciate that their exams are open book and wish to God that the IMO was also open book!

The first corollary includes every radius in the figure; so, if you are given four of them and asked for the fifth, just plug the lengths into the formula. The second, third and fourth corollaries can find all the touching points without having to find any of the centers. To find the centers, it is more accurate to raise perpendiculars from the touching points than to intersect angle bisectors.

It is possible that physical space is Euclidean on a small scale but becomes hyperbolic as distances become greater. In hyperbolic geometry, the sum of the angles in a triangle decrease as size increases – which is why there are no similar triangles except those that are also congruent – but for very small triangles the angle sum is so close to straight that we may not be able to measure its defect with our instruments. But what does "small" mean in the real world? The width of my desk? The width of Germany? The width of the solar system? Thanks to Carl Friedrich Gauss, we know that the triangle with vertices on the peaks of the mountains Hohenhagen, Inselberg and Brocken is small in this sense. We do not know if the world is Euclidean for bigger triangles.

Even if the world is Euclidean all the way out, so excircles always exist, the error in locating an excenter **long** of a large angle is large and approaches infinity as the angle approaches straight. For an engineer, inaccurate tools and the possibility that space is hyperbolic over long distances are indistinguishable; he needs a better method than the excenter theorem to locate excenters.

Let us consider this scenario: The planet Zabol is shrouded in a brown cloud, so civil engineering projects, like a highway interchange, are too big to view from one end to the other. The Zabolians do not know if geometry is hyperbolic on the scale that civil engineers must consider.

They can stretch out a string, pinch it off and walk it somewhere else to replicate a length, and they can rotate it like a compass. But the Zabolian at the other end of the string is invisible in the brown cloud, so optical instruments are useless; and, because of the high winds, there is a limit to how long of strings they can use without the wind causing inaccuracy. Lines are drawn by pounding one stake into the ground at a time. One cannot see more than one stake back, so it is unclear how much inaccuracy each new stake has. For the finding of an excircle to not work means that the Zabolian engineers bisected two exterior angles of a triangle, extended them, and they did not intersect. But they are unsure whether this failure is due to inaccuracy in their stake-by-stake construction of a line, or to their smoggy world having a hyperbolic geometry.

Construction 3.2 *Three highways intersect to make a triangle with sides of given lengths. The highways are connected by arcs of their excircles. Locate the exit ramps to these arcs.*

Solution

By SSS, construct the triangle. By excircle theorem corollary #2, locate the incircle touching points, I_E, I_F, I_G. By excircle theorem corollary #3, locate the excircle touching points on the triangle sides, X_G, Y_F, Z_E. By excircle theorem corollary #4, locate the excircle touching points on the extensions of the sides, $X_E, X_F, Y_E, Y_G, Z_F, Z_G$. ■

Perpendiculars raised from the touching points are as widely spaced on the excircles as possible.

Problem 3.1 *Two country roads intersect at an arbitrary angle. We wish to pave an arc connecting them and going around the corner of a farmer's field, which is on the angle bisector of the two roads.*

In this problem, and later ones about shortcuts, we want the large arc for drivers approaching the intersection and are turning onto the other road, not for those who have driven past the intersection and are making a U-turn to go three quarters around a circle onto the other road.

Solution

Construct a line perpendicular to the angle bisector through the corner of the field. By the center line theorem, this forms an isosceles triangle. Construct the excircle. Find the touching points by dropping perpendiculars from the excenter to the roads. ■

Incredibly, *Common Core* does not consider excircles. How they can ignore the important work of paving companies is beyond me! But now, having helped these brave souls out there laboring over hot asphalt, we are where second semester usually begins, with the first citation of the dreaded parallel postulate. The lemma (*Euclid, Book I, Prop. 27*) is neutral geometry – It was one of the yellow-belt exit exam questions; did you get it? – but the transversal theorem (*Euclid, Book I, Prop. 29*) officially introduces Euclidean geometry.[58] The adventure begins!

Transversal Lemma ***(Euclid, Book I, Prop. 27)***

*If **alternate interior** angles are equal, the two lines crossed by the **transversal** are parallel.*

Proof

Suppose that the two lines are not parallel. Their intersection with each other and with the transversal form a triangle. The two given angles are exterior to and interior to this triangle. Their equality contradicts the exterior angle inequality theorem. ■

[58] *Book I, Prop. 28* is what we will call T & V to avoid making the students memorize the names of angle pairs.

Transversal Theorem[59] ***(Euclid, Book I, Prop. 29)***

If the two lines crossed by a transversal are parallel, then alternate interior angles are equal.

Proof

Suppose $\angle EFG \neq \angle FGH$ are alternate interior angles. Replicate $\angle EFG$ as $\angle FGJ$ with J and H on the same side of $\overleftrightarrow{FG}$. By the transversal lemma, $\overleftrightarrow{EF}$ and $\overleftrightarrow{GJ}$ are parallel. But, by the parallel postulate, there is only one parallel to a line through a point, so H is on $\overleftrightarrow{GJ}$. ■

Transversal Theorem Corollary

Two lines are parallel if and only if a perpendicular to one is perpendicular to the other.

Now that we have the parallel postulate, Saccheri quadrilaterals are right rectangles and, indeed, all rectangles are right rectangles, so the adjective "right" is no longer needed. **T & V** means citing the transversal and vertical angles theorems. We do not name angle pairs. This is in keeping with my policy of not turning geometry into an annoying vocabulary test.

Rectangle Bimedian Theorem

A rectangle's bimedians are equal to the sides they do not cut, and their extensions are parallel.

Pairwise Parallels/Perpendiculars Theorem

If the rays of two angles are pairwise parallel or pairwise perpendicular, then the angles are equal; the only exception is for pairwise perpendicular angles with their vertices inside the other angle, so the angles are supplementary. (This is called quadrilateral angle sum theorem corollary #1.)

Equal Perpendiculars Theorem

Perpendiculars through a point inside a square are equally cut by opposite sides of the square.

Construction 3.3 ***(Euclid, Book I, Prop. 31)***

Construct a line parallel to a given line through a point not on the line.

Solution

Draw an arc around the given point, E, of radius greater than the distance from the line to intersect it at G. Draw an arc of the same radius around G to intersect the given line at F on the E side of the perpendicular at G. Draw an arc of radius $\overline{EF}$ around G and let H be its intersection with the E-arc on the same side of $\overleftrightarrow{FG}$ as E. By SSS, $\overline{EGF} \cong \overline{GEH}$, which holds the equality $\angle EGF = \angle GEH$. By the transversal lemma, $\overleftrightarrow{EH} \parallel \overleftrightarrow{FG}$. ■

[59] Richard Rusczyk writes, "Like the parallel postulate, [the transversal theorem] turns out to be one of those 'obvious' facts that cannot be proved. It must be assumed (2006, p. 26)." Should we tell him?

Construction 3.4 *Construct a line through a point that meets a given line at a given angle.*

This is easy, but it needs to be stated here to make it clear that this construction is Euclidean.

Problem 3.2 *Prove that, if two lines are parallel and a line cuts one of them, it also cuts the other.*

Problem 3.3 *Given $\overline{EFG}$, draw a line through the incenter parallel to $\overleftrightarrow{EF}$ that intersects $\overline{EG}$ and $\overline{FG}$ at J and K, respectively. Prove that $\overline{JK} = \overline{EJ} + \overline{FK}$.*

Problem 3.4 *Given two parallel lines, draw a transversal that cuts one line at an angle such that it is twice (or three times or five times) one of the angles that the other line is cut at.*

For this problem, I hope everybody got the two constructions in the white belt exit exam! Can white belts do this with a given side length? No. But with AA similarity, orange belts can.

Problem 3.5 *You are given two points, a circle, and a line. Draw a circle that passes through the two points and whose common chord with the given circle is parallel to the given line.*

> ***Solution***
> Drop a perpendicular from the center of the circle to the line. By the common chord theorem, the desired circle's center is on this line. By the diameter and chord theorem, the center is on the mediator of the segment connecting the two points. Where these loci intersect is the center. By the transversal theorem corollary, the common chord is parallel to the given line because the line of centers is perpendicular to both. ■

Existence is not assured if the segment joining the two given points is parallel to the given line. Also, the points must be such that the circle cuts the given circle, so there is a common chord.

Angle Sum Theorem ***(Euclid, Book I, Prop. 32)***
Interior angles of a triangle sum to one straight angle; that is, $\alpha + \beta + \gamma = \sigma$.

> ***Proof***
> Given $\overline{EFG}$, by C. 3.3, construct a parallel to $\overleftrightarrow{EF}$ through G. Let J be a point on this line on the same side of $\overleftrightarrow{FG}$ as E. Let K be a point on this line on the same side of $\overleftrightarrow{EG}$ as F.
> $\overleftrightarrow{EG}$ transverses $\overleftrightarrow{EF}$ and $\overleftrightarrow{JK}$. By the transversal theorem, $\angle JGE = \angle GEF$.
> $\overleftrightarrow{FG}$ transverses $\overleftrightarrow{EF}$ and $\overleftrightarrow{JK}$. By the transversal theorem, $\angle FGK = \angle EFG$.
> By supplementarity, $\angle JGE + \angle EGF + \angle FGK$ is straight.
> By substitution, $\angle GEF + \angle EGF + \angle EFG$ is straight. ■

Exterior Angle Theorem ***(Euclid, Book I, Prop. 32)***
An exterior angle equals the sum of the remote interior angles.

Isosceles Angle Theorem
If α is the apex angle of an isosceles triangle, a base angle is $\rho - \frac{1}{2}\alpha$, which is also $\frac{1}{2}(\sigma - \alpha)$. The supplement of the base angle is $\rho + \frac{1}{2}\alpha$, and double the base angle is $\sigma - \alpha$.

ρ is right, and σ is straight; that is, $\rho = \frac{1}{2}\sigma$. We do not have division; $\frac{1}{2}$ is the symbol for bisection. It is just a symbol for C. 1.1; analogy does not give us $\frac{1}{3}$ or any other symbols that look like this.

Quadrilateral Angle Sum Theorem
Interior angles of a quadrilateral sum to two straight angles.

Quadrilateral Angle Sum Theorem Corollaries
1. *If opposite quadrilateral angles are right, then the other two angles are supplementary.*
2. *Let $\overline{EFGH}$ be tangential with incenter I. Then, $\angle EIF + \angle GIH = \sigma = \angle FIG + \angle HIE$.*
3. *Let $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ be tangent to an I-circle and $\overleftrightarrow{FG}$ also tangent. Then, $\angle FIG = \rho$.*

Triangle Centers' Angles Theorem
Let $\overline{EFG}$ have orthocenter H, incenter I and circumcenter O.
1. *If $\angle E < \rho$ and $\angle F < \rho$, then $\angle EHF$ is supplementary to $\angle G$.* $\angle EHF + \angle G = \sigma$
2. *$\angle EIF$ is a right angle plus half of $\angle G$.* $\angle EIF = \rho + \frac{\angle G}{2}$
3. *If $\angle G \leq \rho$, then $\angle EOF$ is double it.* $\angle EOF = 2\angle G$

The orthocenter will be defined soon; I mention it here to get these angle theorems in one place. Parts (1) and (2) are easy corollaries of the angle sum theorem, but they are worth naming. If this is not your first geometry book, you may recognize part (3) as a direct result of the inscribed angle theorem, which green belts will prove. See if you can prove (3) now without looking ahead!

Polygon Angle Sum Theorem
1. *Interior angles of n adjacent triangles sum to n straight angles.*
2. *Exterior angles of n adjacent triangles with a convex union sum to two straight angles.*

Proportions will not be defined until blue belt, but we can define **similar triangles** now as two triangles with all corresponding angles equal. Of course, proportions are a mainstay of standard geometry exams, but you can pass such exams if you know how to cross multiply. Blue belt is not needed, though it helps to look up the intersecting chords and intersecting secants theorems for those annoying exams that cherry-pick some unproven theorems for students to memorize.

Angle–Angle (AA) Similarity Theorem
Two corresponding angles equal is sufficient to prove the similarity of two triangles.

Problem 3.6 *From a house in the country, construct a dirt road to a straight paved road, the latter twice as fast as the former, to minimize travel time to a nearby town on the paved road.*

> ***Solution***
> Guess at where the intersection should be. On the other side of the paved road, construct a half equilateral triangle with the hypotenuse on the paved road to town and the short leg extending into the dirt. From the intersection, you could get to the right vertex or to town in the same time. By definition of segment, if the intersection is not collinear with the house and the right vertex, it is badly guessed. By AA similarity, the driveway must be $\frac{\varphi}{2}$ off the perpendicular from the house to the paved road to make this path straight.■

Pairwise Parallel/Perpendicular Similarity Theorem
If the side extensions of triangles are pairwise parallel or pairwise perpendicular, they are similar.

Problem 3.7 *Given $\overline{EFG}$ with incenter I and excenter X, prove that $\overline{IGE} \sim \overline{FGX}$.*

Problem 3.8 *Prove that, if the bisector of an exterior angle is parallel to the opposite side, then the triangle is isosceles. Is the given angle the base or the apex angle of the isosceles triangle?*

When I was in middle school, I studied art. One day, I drew a horse; I did an exceptional job on the legs – all the joints were hinged in the right direction – but then I ran off the top of the paper, so the horse's head was cut off. Undeterred, I just stapled another sheet of paper to my picture and kept drawing. But graphic artists are not allowed to staple; hence, the following problem:

Problem 3.9 *Two lines meet several centimeters off the paper. Perform these constructions:*
1. *Replicate the angle that they make; and 2. Bisect the angle that they make.*

This is as far as graphic artists usually get, but I recommend the following theorems and P. 3.10.

1. Equal Segments on Parallels Theorem;
2. Parallelogram Theorem;
3. Parallelogram Diagonals Theorem; and
4. Mid–Segment Theorem.

Problem 3.10 *Design a trucker's triangular hazard reflector. Draw an equilateral triangle and then another one with the same center and orientation, but with sides half of the outer lengths.*

Problem 3.11 *Through a point on a circle, draw a chord twice as long as it is from the center.*

Solution
Draw a radius to the point and another perpendicular to it. Connect their endpoints. ■

Proof
Drop a perpendicular from the center to the chord. By HL, the two right triangles it makes are congruent, and, by the angle sum theorem, they are isosceles. ■

If a problem appears in the orange-belt chapter, this means it could not have been solved in the yellow-belt chapter before the parallel postulate was introduced; that is, the solution fails in non-Euclidean geometry. We are not here to teach non-Euclidean geometry, but the textbook was arranged so those who are preparing for a course in non-Euclidean geometry can read to the end of the yellow-belt chapter to learn what is common to our sciences. If they are still with us, it is instructive for them to consider how the solution to a problem fails in non-Euclidean geometry.

If Tokyo (35° 41′ N, 139° 41′ E) is on a circle centered at the north pole, then our "solution" is a point off the coast of California (35° 41′ N, 130° 19′ W). The chord – the great circle that airplanes travel – is 7796 km long. Its midpoint (45° 26.5′ N, 175° 19′ W) is 4955 km from the north pole.

Lambert Theorem
Lambert quadrilaterals (three right angles) are right rectangles.

Proof
Let $\overline{EFGH}$ be a quadrilateral with $\angle E = \angle F = \angle G$, all right. $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ and $\overleftrightarrow{GF} \parallel \overleftrightarrow{EH}$ by the transversal theorem corollary. By the transversal theorem corollary, $\angle H$ is right. ■

The transversal theorem corollary may have seemed too obvious to bother stating; but, without it, this proof would have required repeatedly stating that right angles equal their supplement.

Lambert Theorem Corollary
A parallelogram with at least one right angle is a right rectangle.

Kite: The union of two congruent triangles whose uncommon sides that are equal are also consecutive

Parallelogram: The union of two congruent triangles whose uncommon sides that are equal are also opposite

Kite Theorem
The diagonals of a kite are perpendicular, and the non-definitional diagonal is bisected.

> ***Proof***
> By SAS, there are two congruent triangles on one side of the non-definitional diagonal, so it is bisected. By supplementarity, they are right triangles. ■

Kite Altitudes Theorem
Let $\overline{EFH} \cong \overline{GFH}$*, so* $\overline{EFGH}$ *is a kite. If* H_G, H_E *are pedal triangle vertices in* $\overline{EFG}$*, then* $\overline{H_G F H_E H}$ *is also a kite.*

Except for one kite theorem, orange belts will focus mostly on parallelograms. In parallelogram $\overline{EFGH}$, both $\overline{EFH} \cong \overline{GHF}$ and $\overline{HEG} \cong \overline{FGE}$, so either $\overline{FH}$ or $\overline{EG}$ can be definitional diagonals, unlike kites, which have only one definitional diagonal. For parallelograms, we can choose either.

Viviani[60] Sum Theorem
The altitude to a leg of an isosceles triangle is equal to the sum of the distances to the legs from any point on the base.

> ***Proof***
> Given $\overline{EFG}$ isosceles with P an arbitrary point inside the base, $\overline{EF}$, and P_F, P_E the feet of perpendiculars dropped from P onto $\overrightarrow{EG}$ and $\overrightarrow{FG}$, respectively. $\angle P_F EP = \angle P_E FP$ by the isosceles triangle theorem. Extend $\overrightarrow{P_E P}$ and lay off $\overline{PJ} = \overline{PP_F}$. By the angle sum and vertical angles theorems, $\angle EPP_F = \angle FPP_E = \angle EPJ$. By SAS, $\overline{EPP_F} \cong \overline{EPJ}$, so $\angle EJP$ is right. Let E' be the foot of the altitude to $\overrightarrow{FG}$. By the Lambert theorem, $\overline{EJP_E E'}$ is a right rectangle so $\overline{EE'} = \overline{P_E J} = \overline{PP_E} + \overline{PJ} = \overline{PP_E} + \overline{PP_F}$. By the isosceles altitudes theorem, the other altitude is equal, so $\overline{EE'} = \overline{FF'} = \overline{PP_E} + \overline{PP_F}$. ■

Viviani Similarity Theorem
Viviani triangles *are similar.*

Viviani Difference Theorem
The altitude to a leg of an isosceles triangle is equal to the difference of the distances to the legs from any point on the extension of the base.

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[60] Vincenzo Viviani was a 17th century geometer in Florence, Italy who, with Evangelista Torricelli, were students of Galileo Galilei. With Giovanni Saccheri and Giovanni Ceva, who came fifty years later, this sequence of Jesuits laid the foundation for the 18th century revolution in mathematics ushered in by Euler and other Swiss mathematicians.

Proof

Given $\overline{EFG}$ isosceles with P an arbitrary point on the extension of the base, $\overrightarrow{FE}$. Let P_F, P_E be the feet of perpendiculars dropped from P onto $\overrightarrow{EG}$ and $\overrightarrow{FG}$, respectively. By the isosceles triangle and vertical angles theorems, $\angle P_F EP = \angle P_E FP$. Lay off $\overline{PJ} = \overline{PP_F}$ on $\overrightarrow{PP_E}$. $\angle EPP_F = \angle EPJ$ by the angle sum theorem. $\overline{EPP_F} \cong \overline{EPJ}$ by SAS, so $\angle EJP$ is right. Let E' be the foot of the altitude to $\overrightarrow{FG}$. By the Lambert theorem, $\overline{EJP_E E'}$ is a right rectangle, so $\overline{EE'} = \overline{P_E J} = \overline{PP_E} - \overline{PJ} = \overline{PP_E} - \overline{PP_F}$. Analogously, $\overline{FF'} = \overline{PP_F} - \overline{PP_E}$. In general, $\overline{EE'} = \overline{FF'} = \left|\overline{PP_E} - \overline{PP_F}\right|$. ■

Viviani Equilateral Theorem

The altitude of an equilateral triangle is equal to the sum of the distances to the sides from any point on or inside the triangle.

Proof

Given $\overline{EFG}$ equilateral, let $\overline{GG'}$ be the altitude to $\overline{EF}$ and G'' its intersection to a parallel to $\overleftrightarrow{EF}$ through interior point P. Let P_E, P_F, P_G be its pedal triangle vertices. By equilateral triangle theorem and the Viviani sum theorem, $\overline{PP_E} + \overline{PP_F} = \overline{GG''}$. By the Lambert theorem, $\overline{PP_G G'G''}$ is a right rectangle; thus, $\overline{PP_G} = \overline{G'G''}$. $\overline{GG'} = \overline{GG''} + \overline{G'G''}$. Thus, the conclusion, $\overline{PP_E} + \overline{PP_F} + \overline{PP_G} = \overline{GG'}$. ■

All the Viviani theorems have converses that are true; proofs are left as exercises. The big/small vertex of a triangle is the point that minimizes/maximizes the sum of the distances to the sides. If there are two equally big/small vertices, then the minimum/maximum is anywhere between them, by the Viviani sum theorem. There is no minimum or maximum for an equilateral triangle.

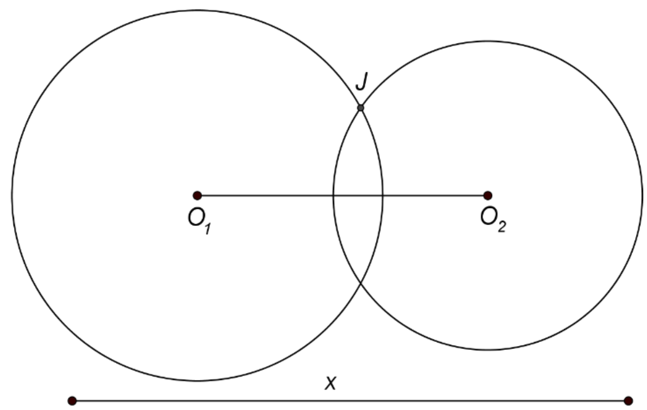
Problem 3.12

Find the locus of points such that the sum of distances to two non-parallel lines is a given length.

Solution

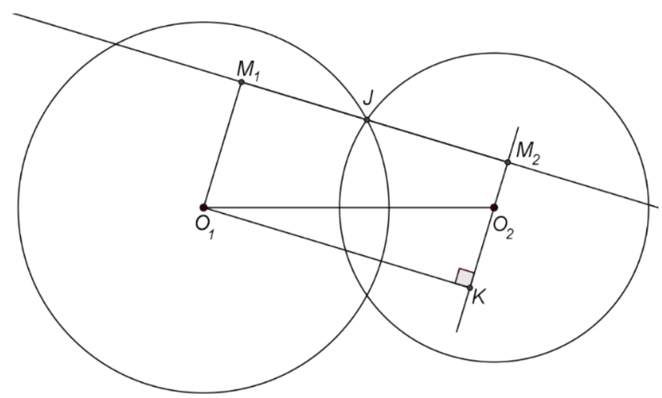
Given lines l_1 and l_2 with intersection O, draw parallels on both sides of l_1 at the given **width**, intersecting l_2 at E and G. If $l_1 \perp l_2$, then $\overline{OE} = \overline{OG}$; else, the perpendicular through O makes two right triangles congruent by AAS, so $\overline{OE} = \overline{OG}$. Lay off $\overline{OE}$ on l_1 to either side of O to F and H; thus, $\overline{EOF}$, $\overline{FOG}$, $\overline{GOH}$, $\overline{HOE}$ are all isosceles triangles whose altitude to a leg is the given length. By the Viviani sum theorem, parallelogram $\overline{EFGH}$ is the desired locus. ■

The locus of differences of distances to two lines that equal a given length is left as an exercise.



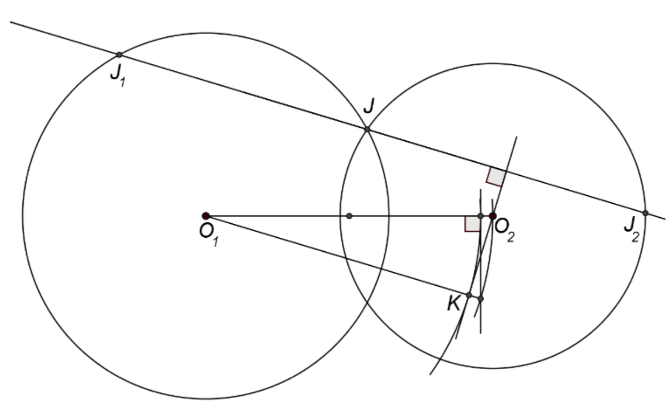
Construction 3.5 *Given two circles with centers O_1 and O_2 that intersect at J, draw a line through J so the distance between its other intersections with the two circles, $\overline{J_1J_2}$, is of a given length, x.*

Note that there are two solutions. If the O_1 and O_2 labels are switched, then $\overline{J_1J_2}$ is tilted downwards on the left.



Guess at $\overline{J_1J_2}$. By C. 1.2, bisect the chords $\overline{J_1J}$ and $\overline{JJ_2}$ to get M_1 and M_2. By the diameter and chord theorem, the diameters through M_1 and M_2 are perpendicular to $\overline{M_1M_2}$. Drop a perpendicular from O_1 onto $\overleftrightarrow{O_2M_2}$ at K and, by the Lambert theorem, $\overline{O_1KM_2M_1}$ is a right rectangle. Because M_1 and M_2 are midpoints, $\overline{M_1M_2} = \overline{O_1K}$ should be half of the given length, x.

It is important to understand that nudging a scratched straightedge around until it fits is only the first step towards the solution. If we stop there, even if the error is too small to see, we are guilty of the trial and error that was denounced in construction 2.1, to trisect an angle. We engage in guesswork only because we can learn from our mistakes when we see how the result fails, not because we hope to succeed. All guesswork leads to failure, but some failures are instructive. C. 2.1 is not; trisecting an angle geometrically is still impossible. But the figure above is helpful! We can bring knowledge (but not measurements) to a clean sheet of paper and do it right.



$\overline{O_1KO_2}$ is right and we have the hypotenuse, $\overline{O_1O_2}$, and a leg, $\overline{O_1K} = \frac{1}{2}x$, so, by HL, it is fully defined. Start over with another photocopy of the original problem and find K by C. 2.2; that is, draw a circle around O_1 of radius $\frac{1}{2}x$, then draw a tangent line through O_2 and call the touching point K. Drop a perpendicular from J onto $\overrightarrow{KO_2}$ and extend it to where it cuts the circles. These points are labeled J_1 and J_2. ■

Problem 3.13 *Draw a line parallel to a given line that cuts off equal chords in two given circles.*

Solution

Given circles ω_1, ω_2, from the center of ω_1, drop a perpendicular on the line. From the center of circle ω_2 drop a perpendicular onto this perpendicular. At its foot, draw ω_3 congruent to ω_2. By the common chord theorem, the line of centers of ω_1 and ω_3 is perpendicular to their common chord. Extend the common chord through ω_2 and drop a perpendicular onto it from the center of ω_2. Three right angles! Lambert steps forward with his theorem and, along with the equal chords theorem, we have our result. ■

Problem 3.14 *Draw a line parallel to a given line that cuts off chords in two given circles such that they have a given sum.*

Solution

Given circles ω_1, ω_2, by C. 3.3 and the Lambert theorem, construct a right rectangle with one pair of sides parallel to the line and vertices at the circle centers. From the center of ω_1, lay off half the given length on the side of the rectangle parallel to the line and raise a perpendicular to the other side of the rectangle. Around this intersection, draw a circle ω_3 congruent to ω_2. Circles intersect circles in none, one or two points and, through any of the intersections of ω_1 and ω_3, draw a line parallel to the given line. ■

Problem 3.15 *Draw a line parallel to a given line that cuts off chords in two circles with a given difference.*

The Lamberts were Calvinists who fled religious persecution and lived in poverty, having left much of their property behind. At twelve, Heinrich Lambert dropped out of school to work full time in his father's tailor shop. At fifteen he took a fulltime job at an ironworks and sent money home to support his younger siblings. But he was never going to be okay with being a semiliterate laborer, and so he studied on his own in the evenings by candlelight. At seventeen he became secretary to a newspaper editor. At twenty he became a tutor to the children of a nobleman, where he made his own astronomical instruments and published in scientific journals. He is known today for Lambert's Law of Absorption and Lambert's Cosine Law, which are about optics, and for theorizing that the stars are not uniformly distributed but clumped in galaxies that rotate around invisible objects of immense mass. We know today that he was the first to make an indirect observation of black holes, which, being black, are known of only by observing their effects on visible matter. Also, he proved π to be irrational and he gave us the Lambert theorem.

Educators assume kids cannot learn if poverty distracts them, but I think wealth is the distraction.

Equal Segments on Parallels Theorem ***(Euclid, Book 1, Prop. 33)***
Connecting the ends of equal segments on two parallel lines forms a parallelogram.

Proof
Let $\overline{EF} = \overline{HG}$ and $\overleftrightarrow{EF} \parallel \overleftrightarrow{HG}$. $\overleftrightarrow{FH}$ is a transversal so $\angle EFH = \angle GHF$ and so $\overline{EFH} \cong \overline{GHF}$ by SAS. ■

Problem 3.16 *A river with parallel banks passes between two towns. Connect the towns with a minimal length road; the bridge must be perpendicular to the river.*

Solution
Let H be your hometown and N be your neighbors. From H drop a perpendicular to the river and lay off $\overline{HE}$ equal to the width of the river. By definition of segment, $\overline{EN}$ is the shortest path from E to N. Let F be the intersection of $\overline{EN}$ with the far side of the river and G be the foot of the perpendicular dropped from F to the near side of the river. By the equal segments on parallels theorem, $\overline{EFGH}$ is a parallelogram, so $\overline{EF} = \overline{GH}$. ■

We know the length of the bridge (it is the width of the river), but we do not know where to position it on the river. So, we construct a line parallel to it that we know how to construct, lay off the known length and then invoke the equal segments on parallels theorem.

Parallelogram Theorem
A quadrilateral is a parallelogram if and only if both pairs of opposite side extensions are parallel.

Part One
If a quadrilateral is a parallelogram, both pairs of opposite side extensions are parallel.

Proof
$\overline{EFH} \cong \overline{GHF}$ holds the equalities $\angle EFH = \angle GHF$ and $\angle EHF = \angle GFH$, which implies that $\overleftrightarrow{EF} \parallel \overleftrightarrow{HG}$ and $\overleftrightarrow{EH} \parallel \overleftrightarrow{FG}$, respectively, by the transversal lemma. •

Part Two
If both pairs of opposite side extensions are parallel, a quadrilateral is a parallelogram.

Proof
Call the intersections E, F, G, H, in this order. $\overleftrightarrow{FH}$ is a transversal to both pairs of parallels, so, by the transversal theorem, $\angle EFH = \angle GHF$ and $\angle EHF = \angle GFH$. $\overline{EFH} \cong \overline{GHF}$ by ASA; so $\overline{EF} = \overline{GH}$ and $\overline{FG} = \overline{HE}$. Thus, by equal segments on parallels. • ■

Subtend–at–Center Theorem ***(Euclid, Book III, Prop. 29)***
Circles are the same or equal if and only if equal chords ***subtend at the center*** *equal angles.*

Proof
If the circles are equal then, by SSS, the triangles formed by radii to the endpoints of equal chords are congruent and so their apex angles are equal. Suppose equal chords $\overline{EF}$ and $\overline{GH}$ subtend at the center equal angles in circles of different radii, r and R, respectively, with $r < R$. Construct them concentrically. $\overline{FG} = R - r = \overline{HE}$ and so, by definition, $\overline{EFGH}$ is a parallelogram because opposite sides are equal. By the parallelogram theorem, $\overleftrightarrow{FG} \parallel \overleftrightarrow{HE}$, a contradiction because radii meet at the center; thus $r = R$. ■

"Equal" is all that we can say about arcs. Length is a magnitude that only applies to line segments. Geometry textbooks that casually refer to the length of arcs are engaged in the unsound practice of inserting calculus results into geometry books and pretending to have proven them. We have no way to measure the length of an arc, but only to know if two arcs are equal. Area is analogous, but we will be able to measure it once we have multiplication; we will never measure arc length.

By the parallels and circle theorem, parallel lines that intersect a circle cut off equal chords between the two lines. By the subtend-at-center theorem, they also cut off equal arcs, which is how this theorem is sometimes written in other textbooks. When those textbooks go beyond discussing equality to assign real numbers to the lengths of arcs, they are not doing geometry anymore, they are doing trigonometry, a subject whose theorems can only be proven with calculus because sine and cosine are defined as infinite series, the work of Calculus II students.

But American students should know that *Common Core* geometry is mostly a review of Algebra I and, when they tire of reviewing the algebraic formula for the length of a segment defined by the Cartesian coordinates of its endpoints, they turn to cross multiplication. This is mostly illustrated with unit conversion (*e.g.,* inches to centimeters) but, to give this algebra exercise the appearance of geometry by drawing a circle, they just *love* the formula $\frac{\theta}{360°} = \frac{s}{2\pi r} = \frac{A}{\pi r^2}$ where θ is angle, s is arc length and A is the area of a sector (like a slice of pie) in a circle of radius r.

Parallelogram Angles Theorem ***(Euclid, Book 1, Prop. 34)***

1. *A quadrilateral is a parallelogram iff both pairs of opposite interior angles are equal.*
2. *A quadrilateral is a parallelogram iff both pairs of opposite exterior angles are equal.*
3. *A quadrilateral is a parallelogram iff any two consecutive angles are supplementary.*

This is easy; modern textbooks spend entirely too much time on these easy theorems.

Parallelogram Diagonals Theorem
A quadrilateral is a parallelogram if and only if the diagonals bisect each other.

Proof

Assume that $\overline{EFGH}$ is a parallelogram. If $\overline{FH}$ is the definitional diagonal, $\overline{EFH} \cong \overline{GHF}$, which holds the equality $\angle EFH = \angle GHF$. If $\overline{EG}$ is the definitional diagonal, $\overline{GEF} \cong \overline{EGH}$, which holds the equality $\angle GEF = \angle EGH$. Let T be the **bi-medial**. By ASA, $\overline{EFT} \cong \overline{GHT}$, which holds the equalities $\overline{FT} = \overline{HT}$ and $\overline{ET} = \overline{GT}$.

Let T be the bi-medial of $\overline{EFGH}$ and assume that $\overline{FT} = \overline{HT}$ and $\overline{ET} = \overline{GT}$. With the vertical angles theorem and SAS, $\overline{EFT} \cong \overline{GHT}$ and $\overline{FGT} \cong \overline{HET}$. Thus, $\overline{EF} = \overline{GH}$ and $\overline{FG} = \overline{HE}$. $\overline{EFGH}$ is a parallelogram by the equal segments on parallels theorem. ■

Right Triangle Median Theorem
The median to the hypotenuse of a right triangle is half of the hypotenuse.

This is an easy corollary of the parallelogram diagonals theorem, and it is often labeled as such.

Mid–Segment Theorem

1. *A mid-segment of a triangle is half the other side, and their extensions are parallel.*
2. *A line parallel to the base of a triangle that bisects one side also bisects the other side.*
3. *Given $\overline{EFG}$, J on the same side of $\overleftrightarrow{GE}$ as F, $\overleftrightarrow{M_{GE}J} \parallel \overleftrightarrow{EF}$ and $\overline{M_{GE}J} = \frac{1}{2}\overline{EF}$, then $J \equiv M_{FG}$.*

Proof

1. Given $\overline{EFG}$, construct a line parallel to $\overleftrightarrow{EG}$ through F and let K be the intersection of it with $\overrightarrow{M_{GE}M_{FG}}$. By the transversal theorem, $\angle GM_{GE}M_{FG} = \angle FKM_{FG}$. With the vertical angles theorem and M_{FG} being a midpoint, $\overline{GM_{GE}M_{FG}} \cong \overline{FKM_{FG}}$ by AAS. By congruence, $\overline{GM_{GE}} = \overline{FK}$. M_{GE} is a midpoint, so $\overline{GM_{GE}} = \overline{M_{GE}E}$. By transitivity, $\overline{M_{GE}E} = \overline{FK}$. By the equal segments on parallels theorem, $\overline{EFKM_{GE}}$ is a parallelogram and $\overline{M_{GE}K} = \overline{EF}$. By the parallelogram theorem, $\overleftrightarrow{M_{GE}M_{FG}} \parallel \overleftrightarrow{EF}$. $\overline{GM_{GE}M_{FG}} \cong \overline{FKM_{FG}}$, so $\overline{M_{GE}M_{FG}} = \frac{1}{2}\overline{M_{GE}K}$. Thus, $\overline{M_{GE}M_{FG}} = \frac{1}{2}\overline{EF}$. •

2. Given $\overline{EFG}$, find J on $\overline{FG}$ such that $\overleftrightarrow{M_{GE}J} \parallel \overleftrightarrow{EF}$. Draw a line through F parallel to $\overleftrightarrow{EG}$ to intersect $\overleftrightarrow{M_{GE}J}$ at K. By the parallelogram theorem, $\overline{EFKM_{GE}}$ is a parallelogram. Thus, $\overline{FK} = \overline{EM_{GE}} = \overline{M_{GE}G}$. By the equal segments on parallels theorem, $\overline{FKGM_{GE}}$ is a parallelogram. By the parallelogram diagonals theorem, $\overline{FJ} = \overline{JG}$. •

3. Extend $\overrightarrow{M_{GE}J}$ that much again to K so $\overline{M_{GE}K} = \overline{EF}$. By the equal segments on parallels theorem, $\overline{EFKM_{GE}}$ is a parallelogram, so $\overline{FK} = \overline{EM_{GE}}$ and $\angle M_{GE}EF = \angle FKM_{GE}$. $\angle M_{GE}EF = \angle GM_{GE}J$ by T&V and $\overline{EM_{GE}} = \overline{M_{GE}G}$; thus, $\angle GM_{GE}J = \angle FKJ$ and $\overline{GM_{GE}} = \overline{FK}$ by transitivity. By SAS, $\overline{GM_{GE}J} \cong \overline{FKJ}$, so $\angle M_{GE}JG = \angle KJF$ and $\overline{GJ} = \overline{FJ}$. By the vertical angles theorem, F, J, G are collinear, so $J \equiv M_{FG}$. • ∎

Medial Triangle Theorem I
*The **medial triangle** is congruent to the three triangles around it and all five triangles are similar.*

Medial Triangle Theorem II
The feet of perpendiculars dropped from a triangle's apex onto its base angle bisectors define a line that is parallel to the base.

Medial Triangle Theorem III
Perpendiculars dropped on interior and exterior angle bisectors from the other vertices of a triangle have their feet on the extensions of the sides of its medial triangle.

Proof
Given $\overline{EFG}$, lay off $\overline{FG}$ on $\overrightarrow{GE}$ to J so $\overline{JFG}$ is isosceles. If $J \equiv E$, then skip to "K is the foot…;" else, by the center line theorem, $\overrightarrow{GM_{JF}}$ bisects $\angle G$. By mid-segment theorem #1 applied to $\overline{EJF}$, $\overleftrightarrow{M_{EF}M_{JF}} \parallel \overleftrightarrow{EJ}$; thus, M_{JF} is on $\overleftrightarrow{M_{EF}M_{FG}}$, the extension of a side of the medial triangle. K is the foot of the perpendicular dropped on the exterior bisector of $\angle G$ from F. By the interior and exterior angles and the Lambert theorem, $\overline{M_{JF}FKG}$ is a rectangle. By the parallelogram diagonals theorem, M_{FG} is also the midpoint of $\overline{M_{JF}K}$; thus K is on $\overleftrightarrow{M_{EF}M_{FG}}$. Analogously for feet of perpendiculars dropped from every vertex on the bisectors of the other vertices. ∎

Problem 3.17 *Prove that the incenter of a triangle lies inside its medial triangle.*

Construction 3.6 *Construct a triangle given the legs and the median to the base.*

Construction 3.7
Construct a triangle given the base, a base angle, and the median to the opposite leg.

Solution
Given $\angle EFG''$ and $\overline{EF}$, the locus of midpoints of $\overline{EG}$ where G is on $\overrightarrow{FG''}$ is the parallel to $\overleftrightarrow{FG''}$ through M_{EF}, by mid-segment theorem #1. Where this locus intersects the locus of points median distant, m_F, from F is M_{GE}. $G := \overrightarrow{EM_{GE}} \cap \overrightarrow{FG''}$. Thus, $\overline{EFG}$. ∎

Construction 3.8 *Given $\overline{G_1G_2} = \overline{G_2G_3}$ on line l_1 and an arbitrary point E_1 on line l_2, find E_2 and E_3 so $\overline{E_1E_2} = \overline{E_2E_3}$.*

This construction works for an arbitrary number of segments, but it is described here for two.

Solution

Given $\overline{G_1G_2} = \overline{G_2G_3}$ on line l_1 and an arbitrary point E_1 on line l_2, Connect $\overline{E_1G_1}$ and draw parallels to it through G_2 and G_3 that intersect l_2 at E_2 and E_3, respectively. If $l_1 \parallel l_2$, then skip to "By the parallelogram theorem;" else, from G_1 draw a parallel to l_2 that intersects $\overline{E_2G_2}$ at F_2. From G_2 draw a parallel to l_2 that intersects $\overline{E_3G_3}$ at F_3. by T & V and ASA, $\overline{G_1G_2F_2} \cong \overline{G_2G_3F_3}$; thus, $\overline{G_1F_2} = \overline{G_2F_3}$. By the parallelogram theorem, $\overline{G_1F_2E_2E_1}$ and $\overline{G_2F_3E_3E_2}$ are parallelograms, so $\overline{E_1E_2} = \overline{E_2E_3}$. ■

Construction 3.9 *Trisect a segment.* ***(Euclid, Book VI, Prop. 10)***

Euclid's Solution

Given $\overline{OE_3}$, draw a ray $\overrightarrow{OG_1}$ and then lay off $\overline{OG_1} = \overline{G_1G_2} = \overline{G_2G_3}$ on it. By C. 3.8, find E_1 and E_2 so $\overline{E_1E_2} = \overline{E_2E_3}$. From G_1 draw a parallel to $\overleftrightarrow{OE_3}$ that intersects $\overline{E_2G_2}$ at F_2. $\overline{OG_1E_1} \cong \overline{G_1G_2F_2}$ by T & V and ASA; thus, $\overline{OE_1} = \overline{G_1F_2}$. By the parallelogram theorem, $\overline{G_1F_2} = \overline{E_1E_2}$. Thus, $\overline{OE_1} = \overline{E_1E_2} = \overline{E_2E_3}$. ■

C. 3.8 is good for cutting a segment into many equal pieces; there are better trisection methods.

Construction 3.10 *Construct a quadrilateral given the four sides and one bimedian.*

Solution

We are given the lengths of the sides in $\overline{EFGH}$ and the bimedian between the midpoints of $\overline{EF}$ and $\overline{GH}$. We guess – draw the figure with the lengths only approximate – and then see what is wrong and how to redraw the figure correctly. Let M_{EF} and M_{GH} be the midpoints of $\overline{EF}$ and $\overline{GH}$. Since we are assuming this figure is the solution, it is fully defined, and we can construct parallelograms $\overline{EJM_{GH}H}$ and $\overline{FGM_{GH}K}$. Connect M_{EF} to the parallelogram vertices, J and K. $\overline{JEM_{EF}} \cong \overline{KFM_{EF}}$ by SAS ($\overline{JE} = \overline{KF}$ and $\overleftrightarrow{JE} \parallel \overleftrightarrow{KF}$, which implies $\angle JEF = \angle KFE$ by the transversal theorem, and $\overline{EM_{EF}} = \overline{FM_{EF}}$ because M_{EF} is a midpoint) which holds the equality $\angle EM_{EF}J = \angle FM_{EF}K$. By the vertical angles theorem, J, M_{EF}, K are collinear. But we guessed at $\overline{EFGH}$, so this is not the solution; the lengths are wrong. So, what have we achieved with this guesswork? $\overline{JKM_{GH}}$ is a triangle! Construct it by C. 3.6, build parallelograms $\overline{EJM_{GH}H}$ and $\overline{FGM_{GH}K}$ and connect $\overline{EF}$. ■

Triangle Frustum Mid–Segment Theorem

*A **triangle frustum**'s **mid-segment** is the **semisum** of the base and the top, and parallel to them.*

Proof

Given $\overline{EFGH}$ with $\overleftrightarrow{EF} \parallel \overleftrightarrow{HG}$, then $\overline{M_{HE}M_{FG}}$ is the mid-segment. Let $J := \overleftrightarrow{EF} \cap \overrightarrow{HM_{FG}}$. By ASA, $\overline{M_{FG}GH} \cong \overline{M_{FG}FJ}$, so $\overline{HM_{FG}} = \overline{JM_{FG}}$ and $\overline{HG} = \overline{FJ}$. By the former equality, $\overline{M_{HE}M_{FG}}$ is the mid-segment of $\overline{EJH}$ and thus $\overleftrightarrow{M_{HE}M_{FG}} \parallel \overleftrightarrow{EF} \parallel \overleftrightarrow{HG}$ and $\overline{M_{HE}M_{FG}} = \frac{1}{2}\overline{EJ}$. But, by the latter equality, $\overline{EJ} = \overline{EF} + \overline{FJ} = \overline{EF} + \overline{HG}$, so $\overline{M_{HE}M_{FG}} = \frac{1}{2}\left(\overline{EF} + \overline{HG}\right)$. ■

Triangle Frustum Mid–Segment Theorem Corollary

*Triangle frustum diagonals cut the mid-segment to the **semidifference** of the top and bottom.*

Proof

Let $P := \overline{M_{HE}M_{FG}} \cap \overline{EG}$ and $Q := \overline{M_{HE}M_{FG}} \cap \overline{FH}$. By the triangle frustum mid-segment theorem, $\overleftrightarrow{M_{HE}Q} \parallel \overleftrightarrow{EF}$. By mid-segment theorems #1 and #2, $\overline{M_{HE}Q} = \frac{1}{2}\overline{EF}$. Analogously, $\overline{M_{HE}P} = \frac{1}{2}\overline{HG}$. Thus, $\overline{PQ} = \left|\overline{M_{HE}Q} - \overline{M_{HE}P}\right| = \frac{1}{2}\left|\overline{EF} - \overline{HG}\right|$. ■

Two Transversals Theorem

Parallel lines that equally cut one transversal equally cut any transversal.

Proof

Let $\overleftrightarrow{H''F} \parallel \overleftrightarrow{HG} \parallel \overleftrightarrow{M''K} \parallel \overleftrightarrow{ML}$ with H'', H, M'', M and F, G, K, L collinear and $\overline{FG} = \overline{KL}$. Find E on $\overleftrightarrow{H''F}$ and J on $\overleftrightarrow{M''K}$ such that $\overleftrightarrow{HE} \parallel \overleftrightarrow{LF} \parallel \overleftrightarrow{MJ}$. By the parallelogram theorem, $\overline{EFGH}$ and $\overline{JKLM}$ are parallelograms, so $\overline{HE} = \overline{MJ}$. By T & V, $\angle EHH'' = \angle JMM''$ and $\angle HEH'' = \angle MJM''$. By ASA, $\overline{HEH''} \cong \overline{MJM''}$, so $\overline{HH''} = \overline{MM''}$. ■

Triangle Frustum Mid–Segment Theorem Converse

A line parallel to the base of a triangle frustum that bisects one leg also bisects the other leg.

This is the two transversals theorem if $\overleftrightarrow{HG}$ and $\overleftrightarrow{M''K}$ are not distinct. We will use contradiction.

Proof

Suppose there is a line parallel to the base through the midpoint of one leg that *does not* bisect the other leg. But, by the triangle frustum mid-segment theorem, the mid-segment is parallel to the base; it *does* bisect the other leg. Two distinct lines through a point not on the base that are parallel to the base contradicts the parallel postulate. ■

Can you prove this independently of the theorem that it is the converse to?

Midpoints and One Altitude Foot Theorem
Triangle side midpoints and the foot of one altitude form an isosceles triangle frustum.

Side–Angle–Side (SAS) Half–Scale Triangle Theorem
If a triangle has two sides that are half the corresponding sides in another triangle and the included angles are equal, then the other angles are equal and the other side also half.

Angle–Side–Angle (ASA) Half–Scale Triangle Theorem
If two pairs of angles are equal in two triangles and the included side of one triangle is half the included side in the other triangle, then the other sides are also half their corresponding sides.

Angle–Angle–Side (AAS) Half–Scale Triangle Theorem
If two pairs of angles are equal in two triangles and a side opposite one of them is half that side in the other triangle, then the other sides are also half their corresponding sides.

Median and Mid–Segment Theorem
The median bisects the mid-segment.

Medial and Parent Triangle Theorem
*The medial triangle and its parent triangle have the same **medial point**.*

Two–to–One Medial Point Theorem
The medial point is unique; it divides each median so the distance from the medial point to the midpoint is half then distance from the medial point to the vertex.

> ***Proof***
> Given $\overline{EFG}$, by mid-segment theorem #1, $\overleftrightarrow{M_{GE}M_{FG}}$ is parallel to $\overleftrightarrow{EF}$ and $\overline{M_{GE}M_{FG}}$ is half of $\overline{EF}$. The median $\overline{EM_{FG}}$ transverses $\overleftrightarrow{M_{GE}M_{FG}}$ and $\overleftrightarrow{EF}$; $\angle FEM_{FG} = \angle M_{GE}M_{FG}E$ by the transversal theorem. The median $\overline{FM_{GE}}$ transverses $\overleftrightarrow{M_{GE}M_{FG}}$ and $\overleftrightarrow{EF}$; by the transversal theorem, $\angle EFM_{GE} = \angle M_{FG}M_{GE}F$. Label the intersection of these two medians C. By the ASA half-scale triangle theorem, $\overline{M_{GE}M_{FG}C}$ is half the lengths of $\overline{FEC}$; thus, C trisects each median as described in the theorem statement. Repeat with another pair of medians. By the uniqueness of magnitudes, this intersection is also C. ■

Problem 3.18 *Prove that the medians' sum is greater than three quarters of the perimeter.*

Problem 3.19 *Given $\overline{EFG}$ and Q the quartile point of $\overline{EG}$ near G, $\overline{QM_{FG}}$ cuts $\overline{GM_{EF}}$ in what ratio?*

Every Triangle a Medial Theorem

Every triangle is medial to some other triangle.

Proof

Given $\overline{EFG}$, by C. 3.3, construct a line through G parallel to $\overleftrightarrow{EF}$, and through F parallel to $\overleftrightarrow{GE}$. Let their intersection be P. By the parallelogram theorem, $\overline{EFPG}$ is a parallelogram and, by the parallelogram diagonals theorem, $\overline{EP}$ and $\overline{FG}$ bisect each other. By C. 3.3, construct a line through E parallel to $\overleftrightarrow{FG}$. Extend $\overrightarrow{PG}$ and $\overrightarrow{PF}$ to intersect it at J and K, respectively. By the median and mid-segment theorem, $E \equiv M_{JK}$. Analogously, $F \equiv M_{PK}$ and $G \equiv M_{JP}$. Thus, $\overline{EFG}$ is medial to $\overline{PJK}$. ■

Orthocenter Theorem

The altitudes are concurrent at a point that we will call the ***orthocenter****.*

Proof

By the every-triangle-a-medial theorem, there exists a parent to the given triangle. The altitudes of the given triangle are the mediators of the parent triangle and, by the circumcenter theorem, they are concurrent. ■

Medial Triangle Orthocenter Theorem

The circumcenter of a triangle is the orthocenter of its medial triangle.

Proof

By mid-segment theorem #1, mid-segment extensions are parallel to their triangle's other sides' extensions. By the transversal theorem corollary, mediators are altitudes of the medial triangle. ■

Half–Scale Orthocenter to Vertex Theorem

The distance from the orthocenter to a vertex of the medial triangle is half the corresponding length in its parent triangle.

Proof

Let $\overline{EFG}$ be a triangle and $\overline{M_{FG}M_{GE}M_{EF}}$ be its medial triangle. Construct altitudes from E and G that intersect at H, the orthocenter of $\overline{EFG}$. Construct altitudes from M_{FG} and M_{EF} that intersect at O, the orthocenter of $\overline{M_{FG}M_{GE}M_{EF}}$. $\angle OM_{EF}M_{FG} = \angle HGE$ and $\angle M_{EF}M_{FG}O = \angle GEH$ by the pairwise parallels theorem. By mid-segment theorem #1, $\overline{M_{EF}M_{FG}}$ is half of $\overline{GE}$. By the ASA half-scale triangle theorem, $\overline{OM_{EF}M_{FG}}$ is half the lengths of $\overline{HGE}$ and so $\overline{OM_{EF}}$ is half of $\overline{HG}$. ■

Problem 3.20 *Given $\overline{EFG}$ with $\angle E = \rho$, let E' be the foot of the perpendicular dropped on $\overline{FG}$ and F_{EG} be the reflection of F. Prove that $\overleftrightarrow{F_{EG}E'} \perp \overleftrightarrow{GM_{EE'}}$.*

Solution

In $\overline{EFE'}$, by mid-segment theorem #1, $\overleftrightarrow{EF} \parallel \overleftrightarrow{M_{EE'}M_{FE'}}$. By the transversal theorem corollary, $\overleftrightarrow{M_{EE'}M_{FE'}} \perp \overleftrightarrow{EG}$. By the orthocenter theorem, $M_{EE'}$ is the orthocenter of $\overline{EM_{FE'}G}$ because it is the intersection of two altitudes; thus, $\overleftrightarrow{GM_{EE'}} \perp \overleftrightarrow{EM_{FE'}}$, because they are the third altitude and third side. In $\overline{F_{EG}E'F}$, by mid-segment theorem #1, $\overleftrightarrow{F_{EG}E'} \parallel \overleftrightarrow{EM_{FE'}}$. By the transversal theorem corollary, $\overleftrightarrow{F_{EG}E'} \perp \overleftrightarrow{GM_{EE'}}$. ■

This is a technique worth remembering: If you are asked to prove two lines perpendicular, then see if they – or another pair of lines that make an equal angle – are an altitude and a side of a triangle. If the other altitudes are perpendicular to their sides, then the desired angle is right!

Problem 3.21 *Given $\overline{EFGH}$ a rectangle and F' the foot of the perpendicular dropped on $\overline{EG}$, prove that $\angle M_{GH}M_{EF'}F = \rho$.*

Let $P := \overline{FF'} \cap \overline{M_{EF'}M'_{EF'}}$ with $M'_{EF'}$ the foot of the perpendicular dropped on $\overline{FG}$. P is the orthocenter of $\overline{M_{EF'}FG}$, so $\overleftrightarrow{GP} \perp \overleftrightarrow{M_{EF'}F}$. Proving $\overleftrightarrow{GP} \parallel \overleftrightarrow{M_{GH}M_{EF'}}$ is left as an exercise, but we again use the orthocenter theorem to prove an angle right and then equate it with another angle.

Problem 3.22 *Given $\overline{EFG}$, build squares on the exteriors of $\overline{EG}$ and $\overline{FG}$ with sides $\overline{EE''}$ and $\overline{FF''}$, respectively. Prove that $P := \overline{EF''} \cap \overline{FE''}$ is on the altitude $\overline{GG'}$.*

Solution

Let E''' and F''' be the feet of perpendiculars dropped on $\overline{FE''}$ and $\overline{EF''}$ from E and F, respectively. If $\overrightarrow{G'G}$ and $\overrightarrow{EE'''}$ and $\overrightarrow{FF'''}$ are concurrent at Q, then $\overline{EFQ}$ is a triangle, $\overline{EF'''}$ and $\overline{FE'''}$ are altitudes, their intersection, P, is the orthocenter of $\overline{EFQ}$ and is on its common altitude with $\overline{EFG}$. Let $J := \overrightarrow{G'G} \cap \overrightarrow{EE'''}$ and $K := \overrightarrow{G'G} \cap \overrightarrow{FF'''}$.

$\angle EFE''' = \angle EJG'$ because both are complementary to $\angle JEF$. $\angle EE''E''' = \angle JEG$ because both are complementary to angles that are equal due to the transversal theorem applied to $\overleftrightarrow{EJ}$ transversing $\overleftrightarrow{EE''}$ and the opposite side of the square on $\overline{EG}$. $\overline{E''E} = \overline{EG}$ because they are sides of this square. Thus, by AAS, $\overline{FE''E} \cong \overline{JEG}$, so $\overline{EF} = \overline{GJ}$. Analogously, $\overline{EF''F} \cong \overline{KFG}$, so $\overline{FE} = \overline{GK}$. Thus, $J \equiv K$. ■

Following are better trisection methods than C. 3.9. But Euclid should not be forgotten; for n not divisible by 2 or 3 (*e.g.,* $n = 5$ or $n = 7$), a modification of Euclid's method is best.

Construction 3.11 *Trisect a segment.*

Medial Point Solution

Draw a line across one end and lay off equal segments so this endpoint of the segment is the midpoint of the base of a triangle and the other endpoint the triangle's apex. Draw in one leg and bisect it. Connect this midpoint to the other end of the base. ■

Proof

By the two-to-one medial point theorem, it trisects the given segment. ■

Indian Solution

Given $\overline{EF}$, draw circles ω_E and ω_F centered at E and F with a radius of about $\frac{1}{4}\overline{EF}$. Draw a segment $\overline{EG}$ that is twice the circle radius and $\angle GEF \approx \rho$. Measure the chord cut by $\overline{EF}$ and $\overline{EG}$ in ω_E and mark the same chord length in ω_F from $\overline{EF} \cap \omega_F$ to J such that J and G are on opposite sides of $\overleftrightarrow{EF}$. $\overline{GJ}$ cuts off a third of $\overline{EF}$. ■

Proof

$K := \overline{EF} \cap \overline{GJ}$. $\angle FEG = \angle EFJ$ because they subtend at the center equal chords in equal circles. By the vertical angles theorem, $\angle EKG = \angle FKJ$, and $\overline{EG} = 2\overline{FJ}$ by construction. By the AAS half-scale triangle theorem, $\overline{EK} = 2\overline{FK}$. ■

Fast–but–Big Solution

Given $\overline{EF}$, draw circles around E and F of radius $\overline{EF}$ and let J and K be their intersecting points. Draw a circle around J with the same radius and let F'' be diametrically opposed to F in this circle. $M := \overline{KF''} \cap \overline{EF}$ cuts off a third of $\overline{EF}$. ■

Proof

$\overline{EFJ}$ and $\overline{FEK}$ are both equilateral and so $\angle EFJ = \angle FEK$. Let $M := \overline{KF''} \cap \overline{EF}$. By the vertical angles theorem, $\angle EMK = \angle FMF''$. $\overline{EK}$ is a radius and $\overline{FF''}$ is a diameter; thus, by the AAS half-scale triangle theorem, $\overline{EMK}$ is half lengths of $\overline{FMF''}$, so $\overline{FM} = 2\overline{EM}$. ■

Class VIII (8th grade, 13 to 14 years old) geometry students in India are taught a six-step method for trisecting a segment, which I call the Indian solution. Six steps is considered par for the course. The fast-but-big solution is a birdie, and the medial point solution bogies. Decide for yourself.

Is there an equally easy means of trisecting an arbitrary angle? Since lengths and angles are both magnitudes, it seems like what goes for one should go for the other. But no, it is impossible, and every geometer should know this of the history of his science. In *Geometry with Multiplication* we will learn that multiplication and division of lengths is defined; of angles, not. *Common Core*, in their rush to turn every geometry problem into an algebra problem, are often seen multiplying one angle by another and getting… something. They just never label their units.

Now let us get back to triangle frustums! First, I will state five basic theorems that everybody knows, but people rarely cite. They will come in handy at the very end of the red-belt chapter.

Isosceles Triangle Frustum Theorem

In an isosceles triangle frustum: (1) base angles are equal; (2) opposite angles are supplementary; (3) legs are equal; (4) diagonals are equal; and (5) the frustum is cyclic. And the converses.

The first four are easy. I feel that *Common Core* textbooks spend entirely too much time on these basic proofs, so I left them as exercises. The fifth is also easy, but it waits for green belts to learn what cyclic quadrilaterals are. We will just not cite the fifth statement until it is proven.

Triangle Frustum Theorem I

Given $\overline{EF}$ with midpoint M_{EF} and E', M'_{EF}, F' the feet of perpendiculars dropped on a line that does not intersect $\overline{EF}$, then $2\overline{M_{EF}M'_{EF}} = \overline{EE'} + \overline{FF'}$.

> ***Proof***
>
> By the transversal theorem corollary and the two transversals theorem, M'_{EF} bisects $\overline{E'F'}$. By the triangle frustum mid-segment or rectangle bimedian theorem, the result by doubling. ■

Triangle Frustum Theorem II

Let E, F, G be collinear, M_{FG} the midpoint of $\overline{FG}$, $2\overline{EF} = \overline{FG}$ and E', F', M'_{FG}, G' be the feet of perpendiculars dropped on a line that does not intersect $\overline{EG}$. Then, $3\overline{FF'} = 2\overline{EE'} + \overline{GG'}$.

> ***Proof***

$2\overline{FF'} = \overline{EE'} + \overline{M_{FG}M'_{FG}}$	Triangle Frustum Theorem I
$2\overline{EE'} + 2\overline{M_{FG}M'_{FG}} = 4\overline{FF'}$	Double
$2\overline{M_{FG}M'_{FG}} = \overline{FF'} + \overline{GG'}$	Triangle Frustum Theorem I
$2\overline{EE'} = 3\overline{FF'} - \overline{GG'}$	Subtract
$3\overline{FF'} = 2\overline{EE'} + \overline{GG'}$	Add $\overline{GG'}$ to both sides ■

Problem 3.23 *Let $\overline{EFG}$ be a right triangle with $\angle EFG$ right and F' the foot of the altitude to the hypotenuse. From F' drop perpendiculars onto $\overline{EF}$ and $\overline{FG}$ with feet F'_G and F'_E, respectively. From F'_G and F'_E drop perpendiculars onto $\overline{EG}$ with feet F''_G and F''_E, respectively. Prove that (1) $\overline{F'F''_G} = \overline{F'F''_E}$; and (2) $\overline{FF'} = \overline{F'_GF''_G} + \overline{F'_EF''_E}$.*

Both are proven by the parallelogram diagonals and triangle frustum mid-segment theorems.

Problem 3.24 *Given $\overline{EFG}$ with midpoints M_{EF}, M_{FG}, M_{GE} and medial point C, let E', F', G', $M'_{EF}, M'_{FG}, M'_{GE}, C'$ be the feet of perpendiculars dropped on a line external to $\overline{EFG}$, respectively; prove that $\overline{EE'} + \overline{FF'} + \overline{GG'} = \overline{M_{EF}M'_{EF}} + \overline{M_{FG}M'_{FG}} + \overline{M_{GE}M'_{GE}} = 3\overline{CC'}$*

> ***Solution***
>
> By triangle frustum theorem I, $2\overline{M_{EF}M'_{EF}} = \overline{EE'} + \overline{FF'}$ and $2\overline{M_{FG}M'_{FG}} = \overline{FF'} + \overline{GG'}$ and $2\overline{M_{GE}M'_{GE}} = \overline{GG'} + \overline{EE'}$. Add these together, collect like terms and halve.
>
> By the two-to-one medial point theorem and by triangle frustum theorem II, we have $3\overline{CC'} = 2\overline{M_{FG}M'_{FG}} + \overline{EE'}$ and $3\overline{CC'} = 2\overline{M_{GE}M'_{GE}} + \overline{FF'}$ and $3\overline{CC'} = 2\overline{M_{EF}M'_{EF}} + \overline{GG'}$. Add these together: $9\overline{CC'} = 2\overline{M_{FG}M'_{FG}} + \overline{EE'} + 2\overline{M_{GE}M'_{GE}} + \overline{FF'} + 2\overline{M_{EF}M'_{EF}} + \overline{GG'}$. But $2\overline{M_{FG}M'_{FG}} = \overline{FF'} + \overline{GG'}$ and $2\overline{M_{GE}M'_{GE}} = \overline{EE'} + \overline{GG'}$ and $2\overline{M_{EF}M'_{EF}} = \overline{EE'} + \overline{FF'}$ by the triangle frustum mid-segment theorem, so $9\overline{CC'} = 3\overline{EE'} + 3\overline{FF'} + 3\overline{GG'}$; dividing through by three, $3\overline{CC'} = \overline{EE'} + \overline{FF'} + \overline{GG'}$. ■

Problem 3.25 *Given $\overline{EFGH}$ a parallelogram and E', F', G', H' the feet of perpendiculars dropped onto a line exterior to the parallelogram, prove that $\overline{EE'} + \overline{GG'} = \overline{FF'} + \overline{HH'}$.*

Let T be the bi-medial of $\overline{EFGH}$ and T' the foot of a perpendicular dropped onto the line. The rest of the proof is just the parallelogram diagonals and triangle frustum mid-segment theorems.

Problem 3.26 *Given $\overline{EFG}$ and E', F', G', M'_{EF} the feet of perpendiculars dropped from E, F, G, M_{EF} onto a line through the medial point C that does not cut $\overline{EF}$; prove $\overline{GG'} = \overline{EE'} + \overline{FF'}$.*

> ***Proof***
>
> By the triangle frustum theorem, $2\overline{M_{EF}M'_{EF}} = \overline{EE'} + \overline{FF'}$. $\overline{CGG'} \sim \overline{CM_{EF}M'_{EF}}$ by AA similarity. By the two-to-one medial point and the ASA half-scale triangle theorems, $\overline{M_{EF}M'_{EF}}$ is half $\overline{GG'}$. Thus, $\overline{GG'} = \overline{EE'} + \overline{FF'}$. ■

Euclidean books define parallel as everywhere equidistant; for us, it is an orange-belt theorem.

Parallel Lines Theorem
Two lines never intersect if and only if they are everywhere equidistant.

Part One
If two lines never intersect, then they are everywhere equidistant.

Proof
Let E and F be arbitrary but distinct points on one line. By C. 1.4, drop perpendiculars to the other line with feet E' and F', respectively. By the parallelogram theorem, $\overline{EFF'E'}$ is a parallelogram and, by definition, opposite sides are equal; hence, $\overline{EE'} = \overline{FF'}$. •

Part Two
If two lines are everywhere equidistant, then they never intersect.

Proof
Suppose they intersect. Connect two points equidistant from the intersection on each line and, that much farther from the intersection, connect two more points. By mid-segment theorem #1, the former is half the latter; the lines are not equidistant. • ■

Construction 3.12 *Construct the two external tangents to two circles of different radii.*

Solution
The radii perpendicular to a tangent have parallel extensions and are of lengths r and $(R-r)+r$; thus, by subtracting r, the tangent is parallel to the tangent from the center of the smaller circle to a circle of radius $(R-r)$ concentric with the larger circle. Draw this tangent by C. 2.2 and then a parallel to it r distant. Do the same on the other side. ■

If an $R-r$ circle is too small to draw, then draw a $2(R-r)$ circle twice as far from the r circle. For best accuracy without going off the paper, start with the tangent and then draw the circles.

Construction 3.13 *Construct the two internal tangents to two disjoint circles of different radii.*

Solution
The radii perpendicular to a tangent have parallel extensions and are of lengths r and $(R+r)+r$; thus, by adding r, the tangent is parallel to the tangent from the center of the smaller circle to a circle of radius $(R+r)$ concentric with the larger circle. Draw this tangent by C. 2.2 and then a parallel to it r distant. Do the same on the other side. ■

Medial Parallelogram Theorem I

Connecting the midpoints of consecutive sides in a quadrilateral form a parallelogram.

> ***Proof***
>
> Construct the mid-segments of the two triangles whose union is the quadrilateral. By mid-segment theorem #1, they are equal, and their extensions are parallel. By the equal segments on parallels theorem, connecting their ends forms a parallelogram. ■

This is called the **medial parallelogram**, or sometimes the Varignon parallelogram after Pierre Varignon, who was French, as were Pierre de Fermat, Henri Pitot, Auguste Miquel, Henri Brocard and Napoleon Bonaparte. All are cited in this textbook and would roll over in their graves to read France's core curriculum, "teut expos'e de logique formalle est exclu" [any formal logic is excluded], or to hear Jean Dieudonné shrieking his slogan, "A bas Euclide! Mort aux triangles!" Such sloganeering is hatred for Gerard Debreu, an economist who spoke long and loud about logic, but used it very badly. There was no reason to throw the baby out with the bath water!

Medial Parallelogram Diagonals Theorem

1. *Medial parallelogram side extensions are parallel to a diagonal of the* ***parent quadrilateral****.*
2. *The* ***perimeter*** *of the medial parallelogram equals the sum of the parent diagonals.*

Easy corollaries of mid-segment theorem #1; the latter is seen in competition problems.

Medial Parallelogram Theorem II

Given $\overline{EFGH}$ not a parallelogram or a triangle frustum, then $M_{FG}, M_{FH}, M_{HE}, M_{EG}$ are the vertices of a parallelogram, as are $M_{EF}, M_{EG}, M_{GH}, M_{FH}$. (The order depends on the shape of $\overline{EFGH}$.)

> ***Proof***
>
> $\overline{M_{HE}M_{FH}}$ and $\overline{M_{EG}M_{FG}}$ are mid-segments of $\overline{EFH}$ and $\overline{EFG}$, respectively, and so both extensions are parallel to $\overleftrightarrow{EF}$ and to each other. $\overline{M_{HE}M_{EG}}$ and $\overline{M_{FH}M_{FG}}$ are mid-segments of $\overline{HGE}$ and $\overline{HGF}$, respectively, and so both extensions are parallel to $\overleftrightarrow{HG}$ and to each other. By the parallelogram theorem, $\overleftrightarrow{M_{HE}M_{FH}} \parallel \overleftrightarrow{M_{EG}M_{FG}}$ and $\overleftrightarrow{M_{HE}M_{EG}} \parallel \overleftrightarrow{M_{FH}M_{FG}}$ imply the first result, with the vertices in some order; the second result is analogous. ■

These are skinny parallelograms and, if $\overline{EFGH}$ is a parallelogram or a triangle frustum, then they are more than just skinny; they are segments because $M_{EG} \equiv M_{FH}$.

Varignon Theorem

The bimedians of a quadrilateral bisect each other.

Be aware that Varignon was also a physicist – there was little difference between physicists and geometers in those days – so, to those taking a class in statics, "Varignon theorem" refers to a physics theorem that he proved. Lambert made his own astronomical instruments; Pitot invented the pitot tube (it measures air speed), and Torricelli invented the barometer. But the most famous physicist of all was Isaac Newton, whose *Principia* was proven with geometry. The header on page 13 reads, "Axioms, or Laws of Motion," and from there on it is all lemmas and theorems written in the style of Euclid's *Elements* and with geometric figures on every page. It is absurd that the editor of the *Real-World Economics Review* writes, "It is a completely mistaken idea that scientific theory is based on deductions from a series of postulates." This outrageous quotation, which appeared in the *American Journal of Islamic Social Sciences*, explains why post-autistic (aka real-world) economists are not welcome in scholarly circles.

Right Triangle Theorem ***(Euclid, Book VI, Prop. 8)***

The altitude to the hypotenuse of a right triangle forms two triangles similar to it and each other.

Proof

AA similarity by the right-angle postulate and by reflexivity. ■

Nested Triangle Theorem

Nested triangles (two transversals that intersect outside two parallel lines) are similar.

Proof

AA similarity by T & V and by reflexivity; or by two applications of T & V. ■

Nested Triangle Theorem Corollary

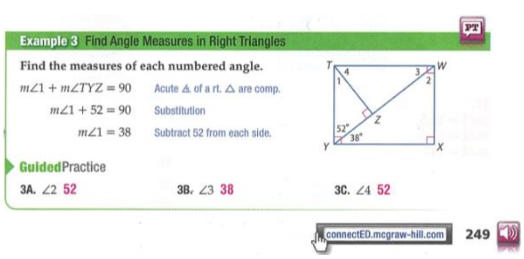
Perpendiculars dropped from points on a ray onto the other ray of an angle form similar triangles.

Crossed Triangle Theorem

Crossed triangles (two transversals that intersect between two parallel lines) are similar.

Proof

AA similarity by T & V; or by two applications of the transversal theorem. ■



Example 3 Find Angle Measures in Right Triangles

Find the measures of each numbered angle.

$m\angle 1 + m\angle TYZ = 90$ Acute ∠s of a rt. △ are comp.

$m\angle 1 + 52 = 90$ Substitution

$m\angle 1 = 38$ Subtract 52 from each side.

GuidedPractice

3A. $\angle 2$ 52 3B. $\angle 3$ 38 3C. $\angle 4$ 52

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This image is from *Glencoe Geometry* and is typical of *Common Core* textbooks. How did it take them 249 pages to get to this simple work? The rest of the page is about a real-world career: personal trainer. There is a picture of a girl stretching! She's hot!

ROFL! Let us lean forward and see if we can accomplish something more daring than that drivel.

Lemma 3.1
A quadrilateral is a rhombus if and only if its diagonals are mediators of each other.

With SSS, this is an easy corollary of the parallelogram diagonals theorem. Rhombi are one of the special parallelograms that were put in the practice problems list for the sake of brevity.

Problem 3.27 *Given rhombus $\overline{EFGH}$ with center C, drop perpendiculars from H, C, G to $\overleftrightarrow{EF}$ at H', C', G', respectively. Prove that $\overline{HC'}$ is perpendicular to the median from E in $\overline{EG'G}$.*

Solution
By the transversal theorem corollary, $\overleftrightarrow{HH'} \parallel \overleftrightarrow{CC'} \parallel \overleftrightarrow{GG'}$. By lemma 3.1, C is the midpoint of $\overline{FH}$; by mid-segment theorem #2 on $\overline{HH'F}$, C' bisects $\overline{H'F}$. $\overline{HC'}$ is median to $\overline{HH'F}$.

$\overline{EG'G} \sim \overline{EC'C} \sim \overline{CC'F} \sim \overline{HH'F}$ The first and last similarities are by the nested triangle theorem. The middle similarity is by the right triangle theorem on $\overline{ECF}$, which we know to be right by lemma 3.1. Thus, $\overline{EG'G} \sim \overline{HH'F}$. Since every side of $\overline{EG'G}$ is perpendicular to the corresponding side of $\overline{HH'F}$, their medians are also perpendicular. ∎

It may seem that kites have been given short shrift. We have proven a lot of interesting and useful theorems about parallelograms; but kites, which are defined as almost the same thing except that the equal sides are consecutive instead of opposite, got only two theorems. Do not worry; as a red belt, you will solve the isosceles kite problem – that is a tough one!

Kite theory finds practical application in tiling. Nobody likes repetitive patterns (*e.g.,* squares or hexagons) on tile floors or wallpaper; it hurts everybody's eyes, and the most sensitive people can go into epileptic fits because the repetitiveness messes with their minds. Thus, much thought has been given to tiling in non-repetitive ways, a field pioneered by Roger Penrose which is today the hottest topic in practical geometry applications.[61] The other Penrose tile, darts, are concave, and are thus not quadrilaterals. We will call them anti-kites to make this distinction clear.

The definitional diagonal bisects the angles at the vertices that it connects. The rhombus is the only quadrilateral that is both a parallelogram and a kite. The square is the only rectangle that is both a parallelogram and a kite. Right rectangles do not exist in non-Euclidean geometry. The most important type of kite in high-school geometry is the **right kite**, which is one in which the two congruent triangles that define it are right triangles. Early in the next chapter, immediately

[61] New 2023 geometry research! Google "einstein tiles" and "vampire einstein" to learn about these inventions.

after the converse to the cyclic quadrilateral theorem, I write, "It is an easy corollary that right kites are cyclic." **Cyclic quadrilaterals** are ones for which a circumcircle exists, and we already know that an incircle does because all kites are tangential.

Quadrilaterals that are both cyclic and tangential are called **bi-centric**. Right kites are not the only bi-centric quadrilaterals, but they are the easiest to construct and, in first-year geometry, probably the only ones that you will encounter. Brahmagupta proved that the area of a cyclic quadrilateral is $\sqrt{(s-e)(s-f)(s-g)(s-h)}$ where e, f, g, h are the lengths of the sides and s is the semiperimeter. Also, he proved that the area of a bi-centric quadrilateral is $\sqrt{efgh} = rs$ where r is the inradius. This is blue-belt geometry, after multiplication is introduced.

Theorems about special parallelograms are in an appendix to this chapter: *Squares and Rectangles and Rombi! Oh My!* What other textbooks call a trapezoid we call a triangle frustum, which is the part of a triangle between the base and a parallel line. This will make sense when we get to solid geometry and learn what a right cone frustum is. The following theorem is in *Practical Shop Mathematics* ([1935] 1958, pp. 178-179) by Wolfe and Phelps, whose stated purpose (p. *v*) was use "not only in factory schools, trade schools, vocational high schools, *etc.*, but also in all high schools to replace the usual geometry course for those students not intending to go to college." *Common Core* eliminated this theorem on the grounds that it is too difficult for their so-called "college ready" students. *Lame!* It makes some Olympiad problems a slam dunk.

Right Triangle Incircle Theorem
A right triangle's indiameter is the sum of the legs minus the hypotenuse.

> ***Proof***
> Given $\overline{EFG}$ with $\angle EFG$ right, by the tangent theorem, $\angle II_GF$ and $\angle II_EF$ are right. By the Lambert theorem, $\overline{II_GFI_E}$ is a right rectangle; indeed, a right square because $\overline{II_G} = \overline{II_E}$. $\overline{I_GF} + \overline{I_EF}$ is the indiameter. By the two tangents theorem, $\overline{EI_G} = \overline{EI_F}$ and $\overline{GI_E} = \overline{GI_F}$. $\overline{EF} + \overline{GF} - \overline{EG} = \overline{EF} + \overline{GF} - \left(\overline{EI_F} + \overline{GI_F}\right) = \overline{EF} + \overline{GF} - \left(\overline{EI_G} + \overline{GI_E}\right) = \overline{I_GF} + \overline{I_EF}$. ∎

Problem 3.28 *Given $\overline{EFG}$ with $\angle EFG$ right and altitude $\overline{FF'}$, prove that the sum of the inradii of the three triangles is $\overline{FF'}$.*

Right Kites in a Right Triangle Theorem
Given $\overline{EFG}$ with $\angle EFG$ right and F' the foot of the altitude to $\overline{EG}$, let J and K be the intersections of the bisectors of $\angle EFF'$ and $\angle GFF'$ with $\overline{EG}$, respectively, and let J' and K' be the feet of perpendiculars from J and K dropped onto $\overline{EF}$ and $\overline{GF}$. $\overline{JJ'} + \overline{KK'}$ is the indiameter of $\overline{EFG}$.

Proof

$2\alpha = \angle F'FE = \angle F'GF$ and $2\beta = \angle F'FG = \angle F'EF$ by the right triangle similarity theorem. By the exterior angle theorem, $\angle EKF = \angle KGF + \angle KFG = \angle EFK = 2\alpha + \beta$. $\overline{EF} = \overline{EK}$ by the isosceles triangle theorem converse. Analogously, $\angle GJF = \alpha + 2\beta$ so $\overline{GF} = \overline{GJ}$. By AAS, $\overline{F'FJ} \cong \overline{J'FJ}$ and $\overline{F'FK} \cong \overline{K'FK}$ ($\overline{F'FJ'J}$ and $\overline{F'FK'K}$ are right kites); thus, $\overline{JF'} = \overline{JJ'}$ and $\overline{KF'} = \overline{KK'}$. By the right triangle incircle theorem, the indiameter of $\overline{EFG}$ is $\overline{EF} + \overline{GF} - \overline{EG} = \overline{EK} + \overline{GJ} - \overline{EG} = \overline{JK} = \overline{JF'} + \overline{KF'} = \overline{JJ'} + \overline{KK'}$. ■

Grasshopper! Are you still with us? You do remember this passage from the introduction, yes?

> *Geometry will be like going back to 1st grade. Sticking segments together end to end or angles together side by side is no more difficult than 1st grade problems about adding chocolates to or subtracting chocolates from a bowl of candies.*

Have the proofs so far not been easy? You have heard of the *No Child Left Behind Act*, right? That law was repealed. In *Geometry–Do*, it is devil take the hindmost! Triangle construction, coming next, will cut enrollment in half. Hopefully, you will be among the elite few who survive.

The following theorem is distinguished by being the only theorem about centroids that can be proven in geometry. *Common Core* textbook authors vainly attempt to dazzle their students with the unproven claim that the medial point is the centroid; that is, the balance point of a triangular metal plate. Charlatans! While true, this can only be proven with calculus. It is pure chicanery to insert this fact here with a wave of the hand and a boastful smirk.

Thus, in this textbook, we do not use the term centroid to refer to the medial point as other geometry textbooks do. Vague hand-waving references to advanced mathematics is what killed *New Math* in the 1970s and is what is currently killing *Common Core*. It is important that students understand that everything in this book is not just true, but is proven here, within our system.

The only thing we borrow from physics is the teeter-totter principal: A segment of uniform mass subject only to gravity and supported at only one point balances iff that point is the center.

Parallelogram Centroid Theorem

The bi-medial point of a parallelogram is its ***centroid****.*

Proof

Let J and K be points on $\overline{EF}$ and $\overline{GH}$, respectively, and collinear with the bi-medial, C, of parallelogram $\overline{EFGH}$. By ASA, $\overline{ECJ} \cong \overline{GCK}$, which holds the equality $\overline{CJ} = \overline{CK}$. ■

Triangle construction is the province of green belts; but, just to let those intermediate geometers know that they are not really all that special, we will here construct some triangles.

Green belts will learn that the first step to constructing a triangle is to form a hypothesis about what theorem is relevant; then they test their hypothesis by attempting to draw a figure that makes use of the theorem. For instance, construction 3.6 gives the legs and the median to the base. Medians bisect segments. The parallelogram diagonals theorem, which was proven on the previous page, is one of only two theorems about segments being bisected. Big hint!

Here we will make it easy for orange belts by just telling them: The excircle theorem and its corollaries are relevant. Green belts will not get hints. They must scan the index looking for theorems that seem relevant to the triangle construction that they are tasked with solving.

Construction 3.14
Construct a triangle given its semiperimeter, its apex angle and its apex angle bisector.

> ***Solution***
> By the excircle theorem, $\overline{GX_E}$ is the semiperimeter of $\overline{EFG}$; thus, by ASA, $\overline{GX_EX}$ is fully defined. Construct it. Draw the excircle ω_X and $\overline{GX_F}$. Lay off the apex angle bisector on $\overrightarrow{GX}$ and label this point P. By C. 2.2, draw a line through P tangent to ω_X. It cuts $\overline{GX_E}$ at E and $\overline{GX_F}$ at F. ■

Whenever you construct a triangle, you must discuss how many possible solutions there are. Here, C. 2.2 requires that P be outside ω_X; so, if the apex angle bisector is long enough that it is inside or past ω_X, then there is no solution. If P is on ω_X, then there is one solution. If the apex angle bisector falls short of ω_X, so P is outside ω_X, then there are two solutions.

Construction 3.15
Construct a triangle given its base, its apex angle and its inradius.

> ***Solution***
> By AAS, $\overline{GI_FI}$ is fully defined. Construct it. Draw the incircle ω_I and $\overrightarrow{GI_E}$. By the excircle theorem and its second corollary, $\overline{GX_E} = s = \overline{GI_F} + \overline{EF}$. Lay this off on $\overrightarrow{GI_F}$. By C. 1.3, raise a perpendicular from X_E; where it crosses $\overrightarrow{GI}$ is X. Draw ω_X. By C. 3.13, construct the internal tangent to ω_I and ω_X; it cuts $\overrightarrow{GI_F}$ at E and $\overrightarrow{GI_E}$ at F. ■

ω_I and ω_X must be disjoint; if they touch, there is only one solution, otherwise there are two.

Construction 3.16

Construct a triangle given its inradius, its apex angle and the sum of its legs.

Solution

By AAS, $\overline{GI_F I}$ is fully defined. Construct it and measure $\overline{GI_F}$. By the second excircle theorem corollary, $\overline{GI_F} = s - \overline{EF} = \frac{1}{2}\left(\overline{EF} + \overline{FG} + \overline{GE}\right) - \overline{EF} = \frac{1}{2}\left(\overline{FG} + \overline{GE} - \overline{EF}\right)$. But $\overline{GI_F}$ is known and $\overline{FG} + \overline{GE}$ is given, so $\overline{EF} = \overline{FG} + \overline{GE} - 2\overline{GI_F}$ fully defines $\overline{EF}$. Thus, the problem is reduced to C. 3.15. ■

$2\overline{GI_F} < \overline{FG} + \overline{GE}$ must be true for $\overline{EF}$ to exist, which is necessary. But, even if $\overline{EF}$ exists, ω_I and ω_X must be disjoint; if they touch, there is only one solution, otherwise there are two.

Construction 3.17

Construct a triangle given its inradius, a base angle, the difference of its legs, and which is longer.

Solution

Given $\angle E$, by AAS, $\overline{EI_G I}$ is fully defined. Construct it and draw the incircle, ω_I. By the third excircle theorem corollary, $\overline{I_G X_G}$ is the absolute difference of $\overline{EG}$ and $\overline{FG}$. By the incircle and excircle theorem corollary, M_{EF} bisects $\overline{I_G X_G}$. There are three cases:

1. If $\overline{EG} = \overline{FG}$, then $M_{EF} \equiv I_G$, so M_{EF} is located.
2. If $\overline{EG} < \overline{FG}$, then extend $\overline{EI_G}$ by $\frac{1}{2}\overline{I_G X_G}$ past I_G to locate M_{EF}.
3. If $\overline{FG} < \overline{EG}$, then shorten $\overline{EI_G}$ by $\frac{1}{2}\overline{I_G X_G}$ off the I_G end to locate M_{EF}.

Double $\overline{EM_{EF}}$ to get $\overline{EF}$. Drop tangents to ω_I from E and F; they intersect at G. ■

Construction 3.18

Construct a triangle given its inradius, the altitude to one leg and the difference of its legs.

Solution

By the third excircle theorem corollary, $\overline{I_G X_G} = \left|\overline{FG} - \overline{EG}\right|$. By SAS, $\overline{II_G X_G}$ is fully defined. By C. 1.2, bisect $\overline{I_G X_G}$. This midpoint is M_{EF} by the incircle and excircle theorem corollary. If $\overline{EE'}$ is the altitude we are given, then let M'_{EF} be the foot of a perpendicular dropped onto $\overleftrightarrow{FG}$ from M_{EF}. We cannot draw it without $\overline{FG}$; but, by the transversal theorem corollary and mid-segment theorems #2 and #1, $\overline{M_{EF}M'_{EF}} = \frac{1}{2}\overline{EE'}$. Draw a circle of this radius around M_{EF}; also draw the incircle. By C. 3.12, draw an external tangent to these circles; it cuts the extension of $\overline{I_G X_G}$ at F. Double $\overline{M_{EF}F}$ to find E. By C. 2.2, find G. ■

Orange Belt Exit Exam

Four Feet on Angle Bisectors Theorem
The feet of the perpendiculars dropped from the apex of a triangle onto the bisectors of the interior and exterior base angles are collinear.

Inscribed Octagon Theorem
Given a square with circles around each vertex of radii equal to half the diagonal, the circles cut the square at the vertices of a regular octagon.

Parallelogram Angle Bisectors Theorem
Given a parallelogram that is not a square, its angle bisectors form a rectangle.

1. *Three roads,* $\overleftrightarrow{EF}$, $\overrightarrow{EG}$, $\overrightarrow{FG}$, *would make a triangle,* $\overline{EFG}$, *with vertices* E *at* $(-120,-110)$, F *at* $(90,-160)$ *and* G *at* $(0,240)$, *in meters. But, instead of making sharp turns at the vertices,* $\overleftrightarrow{EF}$ *will have exits to curve into* $\overrightarrow{EG}$ *and* $\overrightarrow{FG}$ *that are arcs of the* ω_Z *and* ω_Y *excircles. What is the distance between the exits to roads* $\overrightarrow{EG}$ *and* $\overrightarrow{FG}$ *on road* $\overleftrightarrow{EF}$*?*

2. *A right triangle has sides of* 8 *cm,* 15 *cm and* 17 *cm. What is the sum of the indiameter and the circumdiameter?*

3. *Two points on a segment or its extension are an* **isotomic conjugate** *if they are equidistant from the segment midpoint. Prove that* Y_E *and* Z_F *are isotomic conjugates relative to* $\overline{EF}$.

4. *Given a segment, construct another segment that is one fifth as long.*

5. *Construct a triangle given its perimeter, its apex angle, and its apex altitude.*

6. *Given a parallelogram* $\overline{EFGH}$ *that is not a rhombus, draw a ray from* E *through* $\overline{GH}$ *at* J *and through* $\overrightarrow{FG}$ *at* K. *Prove that (1)* $\overline{EFK} \sim \overline{JHE} \sim \overline{JGK}$*; and (2)* $\angle E$ *is bisected iff* $\overline{JG} = \overline{KG}$.

7. *Given* $\overline{EFG}$ *and parallelogram* $\overline{EJLK}$ *with* J *inside* $\overline{EF}$, K *inside* $\overline{EG}$ *and* L *long of* $\angle E$ *(past* $\overline{FG}$*), let* $M := \overline{FG} \cap \overline{JL}$ *and* $N := \overline{FG} \cap \overline{KL}$. *Prove that* $\overline{EFG} \sim \overline{JFM} \sim \overline{LNM} \sim \overline{KNG}$.

8. *Given two circles, a line, and a length, construct a line parallel to the given one so it cuts the two given circles the given length apart. How many possible solutions are there?*

9. *Given* $\overline{EFG}$ *with medians* m_E, m_F, m_G, *construct a triangle whose sides are these lengths and prove that its medians are three quarters the lengths of the sides of* $\overline{EFG}$.

Practice Problems

3.29 *Mathematical induction means proving a statement $P(n)$ true for every positive integer by first proving it true for $n = 1$ and then proving that, if $P(k)$ is true for some integer k, then $P(k+1)$ follows. An example is proving that the sum of all the integers from 1 to n is $\frac{n(n+1)}{2}$. Prove the polygon angle sum theorem using mathematical induction.*

3.30 *Given $\overline{EFG}$ with orthocenter H and orthic triangle $\overline{E'F'G'}$, prove the following:*

1. *If $\overline{E'F'G'}$ is also the orthic triangle of $\overline{HFG}$, $\overline{EHG}$ and $\overline{EFH}$.*
2. *E, F, G are the orthocenters of $\overline{HFG}, \overline{EHG}, \overline{EFH}$, respectively.*

3.31 *Prove that the distance from the midpoint of the base to the circumcenter is half the distance from the apex to the orthocenter.*

3.32 *Given $\overline{EFG}$ with $e = 8$ and $f = 15$ and $g = 17$, what are the following lengths?*

1. *The distance from G to the touching points of ω_I.*
2. *The distance from G to the touching points of ω_X.*
3. *The distance between where ω_I and ω_X touch $\overleftrightarrow{EF}$.*
4. *The distance between where ω_I and ω_X touch $\overleftrightarrow{EG}$ or $\overleftrightarrow{FG}$.*
5. *The distance between where ω_Y and ω_Z touch $\overleftrightarrow{EF}$.*
6. *The distance between where ω_Y and ω_Z touch $\overleftrightarrow{EG}$ or $\overleftrightarrow{FG}$.*

3.33 *Prove the theorems about special quadrilaterals that were left unproven in the text:*

1. *Rhombus diagonals are perpendicular. (The converse is not necessarily true.)*
2. *Rectangle diagonals are equal. (The converse is not necessarily true.)*
3. *A quadrilateral is a rectangle if and only if all its angles are right.*
4. *A parallelogram is a rectangle if and only if its diagonals are equal.*
5. *A parallelogram is a rhombus if and only if its diagonals bisect the vertex angles.*
6. *The diagonals of a triangle frustum are equal if and only if the triangle is isosceles.*

3.34 *Draw O_1 and O_2 circles of radii $r_1 < r_2$ such that (1) $\overline{O_1O_2} < r_2 - r_1$; (2) $\overline{O_1O_2} = r_2 - r_1$; (3) $r_2 - r_1 < \overline{O_1O_2} < r_2 + r_1$; (4) $\overline{O_1O_2} = r_2 + r_1$; and (5) $\overline{O_1O_2} > r_2 + r_1$.*

3.35 *Inscribe a rectangle in a right triangle so the sum of the diagonals is minimal.*

3.36 *Prove that any one angle and any one side fully define an isosceles triangle.*

3.37 *Prove that, if one interior angle of a parallelogram is right, then they all are.*

3.38 *Two circles are tangent externally at P and have an external tangent that touches them at E and F. Prove that $\angle EPF = \rho$. Note that this proves Thales' diameter theorem.*

3.39 *The extensions of two radii of a circle, at a right angle to each other, are cut at E and F by a line tangent at P. Prove that the other tangents from E and F are parallel.*

3.40 *The altitude of an equilateral triangle is equal to the sum of the distances to the legs minus the distance to the base from any point in the apex's field of fire past the base.*

3.41 *Given $\overline{EFG}$ with base $\overline{EF}$ and E', F' the feet of altitudes, prove that $\overline{EE'G} \sim \overline{FF'G}$.*

3.42 *Given $\overline{EFG}$ isosceles with base $\overline{EF}$ and E', F' the feet of altitudes, prove that $\overline{EFG} \sim \overline{E'F'G}$.*

3.43 *Given a right triangle frustum such that a diagonal is perpendicular to a leg, prove the triangles it makes are similar.*

3.44 *Two rivers, each with parallel banks, but not to each other, pass between two towns. Minimize the road between the towns; the bridges must be perpendicular to the rivers.*

3.45 *The intersection of three streets makes a triangle with sides 50 m, 120 m, and 130 m. What is the turning radius of a traffic circle inside this triangle?*

3.46 *Two country roads intersect at an arbitrary angle. We wish to pave an arc connecting them; use the turning radius mandated by the Highway Department.*

3.47 *To draw a* ***trefoil window****, construct an equilateral triangle and circumscribe it. At each vertex, draw an arc of the same radius as the circumcircle and outside it. If this window is to fit in a wall with studs 16" apart, draw detailed plans for framing it.*

3.48 *Construct a triangle given the base, the apex altitude and (1) one base angle; (2) one leg; or (3) the median to the base.*

3.49 *The inradius of an equilateral triangle is half its circumradius; also, an equilateral triangle with a given incircle has sides twice the length of one with that circumcircle.*

3.50 *Given a circle and a triangle, circumscribe a similar triangle around the circle.*

3.51 *Connect two circles with a segment parallel to and of the same length as a given segment.*

3.52 *Construct a square so that four given points are each on a different side or its extension.*

Orange Belt Geometry for Construction Workers, Revisited

I moved *Orange Belt Geometry for Construction Workers* to the end of the yellow-belt chapter because, if this textbook is to be used for general students and not just in honors classes, then it must help carpenters and masons excel in their chosen trade. But few construction workers survive orange belt,[62] so I moved the section to where they will find it while they are still with us.

> *Arches with one center are called Roman and their chord is called the spring line. A Gothic arch has two centers; if they are at the endpoints of the spring line, it is called equilateral. If the centers are on the extensions of the spring line, it is called lancet; if inside, it is called deep.*

By the chord inside circle theorem, Roman arches exist in a non-Euclidean world if you start with the circle; but, if you start with the rectangle, then they do not. By the chord inside circle and the triangle inequality theorem, Gothic arches and Tudor bridges exist. Tudor arches exist; because the line of centers is less than double the crown arcs' radii. But the isosceles triangle frustums that the individual stones are cut into do not exist. The ogee arch does not exist because proof that the haunch and crown arcs touch cites the parallelogram diagonals theorem. Most arches are facades that must adapt to an existing structure, so situations like this are typical:

> *Suppose that a beam bridge made of reinforced concrete spans a canal four meters wide and it is h meters above the concrete sides of the canal. This is ugly, so the city has hired you, a mason, to construct a façade to make it appear that the bridge is a Tudor arch made entirely of brick.*

Parallel means "lines that do not intersect." In hyperbolic geometry, a line and a point not on it do *not* fully define the parallel through it. Recall C. 2.5; let $Q \equiv M_{M_{EF}F}$ so the subscripts do not stack up. $\angle KLM_{QL} > \rho$, but $\overleftrightarrow{KL}$ *may* be parallel to the mediator of $\overline{QL}$, despite being angled towards it. There is an **angle of parallelism** that depends on how far L is from the mediator of $\overline{QL}$. Only inside this angle do rays intersect the line. The farther the point is from the line, the smaller the angle of parallelism. If $x = \overline{LM_{QL}}$ is this distance, then the angle of parallelism is $2\,\mathrm{atan}(e^{-x})$. O exists for C. 2.5 if the height is greater than a quarter of the width; but, in hyperbolic geometry, O may exist only for arches significantly taller than a quarter of their width, the limit being determined by the defect of triangles that size. This is why Gauss measured the angle sum of a triangle with vertices on mountain peaks. If he had found a defect, then there would be a limit to big squat arches; a London-to-Paris arch may need to be taller than 86 km.

[62] If you are a construction worker and are still with us, then hurray for you! (I have worked in construction too.)

The Cocktail Party Explanation of Non-Euclidean Geometry

Incredibly, nobody at a cocktail party has ever asked me for advice on how they might set an ambush with heavy machine guns, which is what green-belt *Geometry–Do* is mostly about. Instead, I wind up fielding questions about non-Euclidean geometry, which the people apparently learned of in science-fiction novels. In this section, I explain how to reply to such questions.

Bernhard Riemann invented elliptic geometry, which is about geometric figures drawn on the surface of an ellipsoid. The simplest ellipsoid is a sphere, and this geometry has great practical application to navigation on Earth. A line is the path that circumnavigates the globe and there all lines intersect each other. This has been extended to higher dimensions. If you are asked if, given a powerful enough telescope, you would see the back of your own head, the simple answer is no. Gravity can bend light, and a black hole can bend it into a circle or, more likely, a spiral; but black holes are not what people are referring to when they ask this question.

Gauss measured the angle sum of a triangle with vertices on three mountain peaks. This is a famous story, though some of the people telling it seem to have no idea what he was doing up there; I have heard everything from proving that the world is round to proving General Relativity. He was on mountain peaks because he knew that the world is round; he was trying to get line of sight over great distances. General Relativity had not been invented yet, but it would not have affected his experiment because the Earth does not have enough gravity to noticeably bend light.

There are many ways to prove that the Earth is round. This has been known for a long time. If Columbus had just asked, 15th century scientists could have estimated its diameter to within a few hundred klicks, which would have convinced him that he could not reach India by sailing westward. But he knew something was out there and within range of his ships because, after a storm, seeds from American trees wash ashore on European beaches. The triangle drawn on a globe between distant ports has an angle sum greater than straight, but this is not a good method of estimating the Earth's diameter because neither ships nor airplanes make a dead reckoning for their destination, the former because of ocean currents and the latter because the Earth rotates underneath them. This is why those scary maps with circles around North Korea to indicate who is within range of their missiles are wrong. It is actually easier for the North Koreans to launch straight up, wait for the Earth to rotate and then drop their bomb on our East Coast than it is to launch at a low angle and try to overtake the spinning Earth to hit our West Coast.

As discussed in the previous section, Gauss was testing the hypothesis that space is only locally Euclidean and is hyperbolic at longer distances. Germany is not big enough, but it is still possible that, if we did this with telescopes at Earth, Jupiter, and Saturn, we might find a defect.

If asked for the *Geometry–Do* perspective on non-Euclidean geometry, the best response is to state some theorems that you know and contrast them with hyperbolic geometry theorems.

Euclidean Geometry	Hyperbolic Geometry
Circumcenter Theorem The mediators are concurrent at a point equidistant from the vertices.	The circumcircle may not exist. Because the angles sum is less than a straight angle, the mediators of the sides may not intersect.
Transversal Theorem If the two lines crossed by a transversal are parallel, alternate interior angles are equal.	There are many lines through a point on a transversal that do not intersect the line. Alternate interior angles may not be equal.
Angle Sum Theorem Interior angles of a triangle sum to one straight angle; that is, $\alpha+\beta+\gamma=\sigma$.	Interior angles of a triangle sum to less than a straight angle; the larger the triangle, the greater the defect, $\sigma-(\alpha+\beta+\gamma)$.
Angle–Angle (AA) Similarity Theorem Two corresponding angles equal is sufficient to prove the similarity of two triangles.	Only congruent triangles are similar; a triangle with proportionally longer sides has smaller angles and is a distortion of the given triangle.
Lambert Theorem Lambert quadrilaterals (three right angles) are right rectangles; all four angles are right.	Right rectangles do not exist. All rectangles are acute rectangles; four equal acute angles.
Triangle Area Theorem Triangles with equal collinear bases and apexes on a line parallel to their bases are of equal area.	Triangles have equal area if and only if the sum of their interior angles are equal; that is, a triangle's area is proportional to its defect.

Arthur Beiser (1987, p. 149) writes, "The requirement that quantum physics gives the same results as classical physics in the limit of large quantum numbers was called by Bohr the *correspondence principle*." For example, "When $n=2$, [quantum physics] predicts a radiation frequency that differs from [classical physics] by almost 300%. When $n=10{,}000$, the discrepancy is only about 0.01%."

There is no direct relation between the two sciences, but the analogy is that Euclidean geometry is the boundary case of hyperbolic geometry as triangles get smaller in the same way that Newtonian physics is the boundary case of quantum mechanics as quantum numbers get larger. Thus, Gauss observed triangles on mountain peaks and Bohr observed hydrogen atoms.

What is Known of Triangles with an Inaccurate Apex?

Given the base and the two base angles, by ASA, the triangle is fully defined. But what if its apex is off the edge of the paper? Scaling it down by bisecting or trisecting the base might get the whole triangle on the paper, but this is not a complete solution because it is now twice or three times as difficult to locate points related to the triangle, such as its medial point. Getting a bigger sheet of paper helps,[63] but only a little. A compass with a beam can draw circles with radii up to 24 cm, but most student compasses only reach 12 cm and a carpenter's trammel is a bit clumsy. Also, poster-size paper is expensive; plus, your mom needs the kitchen table.

Most people consider Euclid's parallel postulate to be obvious and disparage Lobachevski for inventing a science that is just "playing with axioms" and clearly of no use to any practical man.[64] But we consider the angle sum theorem to be obvious only because we have good eyesight and can see from one mountaintop to another, as Gauss did in his famous experiment. At the beginning of this chapter, I imagine a planet Zabol shrouded in a brown cloud so opaque that a surveyor cannot see the Zabolian pulling on the other end of a surveyor's chain. They do not know if Zabol is hyperbolic on the scale of civil engineering projects. They trust in neutral geometry and denounce Euclid as the one playing with axioms that have no empirical verification.

Geometry on Zabol is a productive thought experiment[65] because the principal motivation for our science is that we can teach lesser scientists (*e.g.,* economists) not to make tacit assumptions. But, if we are going to teach this lesson, we must first be sure that we have learned it ourselves. We trust in Euclid because we have good eyesight. If pigs were the intelligent species on Earth, they would never have conducted the experiment that Gauss did because they can barely see a meter past their flat noses. They would not put their trust in Euclid like the gullible humans do.

To answer the question at the head of this section, we can accurately locate M_{EG} and M_{FG} with the mid-segment theorem even if G is off the paper; draw lines through M_{EF} parallel to $\overleftrightarrow{FG}$ and $\overleftrightarrow{EG}$ and where they intersect $\overline{EG}$ and $\overline{FG}$ are midpoints. $C := \overline{EM_{FG}} \cap \overline{FM_{EG}}$ is accurate, and the circumcenter, O, is accurately found at the intersection of perpendiculars raised from M_{EG} and M_{FG}. We can drop perpendiculars from M_{EG} and M_{FG} onto $\overleftrightarrow{EF}$ and know that G is on the line parallel to $\overleftrightarrow{EF}$ and at twice this width, and we know that it is on the circumcircle. This does not locate G exactly, but it provides quite a bit more information about it than we had before.

[63] This is why *Geometry–Do* is printed on U.S. letter-size paper, even in countries where A4 is the standard size.
[64] People said the same thing about complex analysis – until it became the foundation for electrical engineering.
[65] What interests me about exoplanets is not the prospect of ever visiting or even communicating with aliens – the distances are too great – but conducting thought experiments about what alien ecology is like in a different climate or with more carbon, what astronomy is like with no view of the sky, what ballistics is like with twice the gravity, *etc.*

"Translate" and "Rotate" Are Not Magic Spells

Squares on Rectangles Theorem

On the sides of a rectangle, $\overline{EFGH}$, squares are constructed, lying exterior to it. Their centers, $C_{EF}, C_{FG}, C_{GH}, C_{HE}$, are themselves the vertices of a square.

This theorem is easy; proof is left as an exercise. *Common Core* proponents prove it, pronounce "rotate" like a magic spell and, with a wave of their hand, claim to have proven the following:

Lemma 3.2

1. *The bi-medial point of a square is the vertex of right angles to the corners.*
2. *A rhombus with one right angle is a right square.*

Thébault Theorem

On the sides of a parallelogram that is not a rectangle, $\overline{EFGH}$, squares are constructed, lying exterior to it. Their centers, $C_{EF}, C_{FG}, C_{GH}, C_{HE}$, are themselves the vertices of a square.

Proof

Let $\alpha = \angle EFG = \angle EHG < \angle FEH = \angle FGH = \beta$; if the inequality is backwards, switch the labels of the points. By addition, $\angle C_{EF}FC_{FG} = \angle C_{GH}HC_{HE}$ because both are half a right angle plus half a right angle plus α. Let γ be the angle between the squares with vertex E. All the angles around E must sum to two straight angles and they are two right angles β and γ; thus, β and γ are supplementary. But, by the parallelogram angles theorem, α and β are supplementary, so $\alpha = \gamma$ and we can re-label γ as α. By SAS, $\overline{C_{EF}EC_{HE}} \cong \overline{C_{EF}FC_{FG}} \cong \overline{C_{GH}GC_{FG}} \cong \overline{C_{GH}HC_{HE}}$, all with half the big diagonal and half the small diagonal making an angle of half a right angle plus half a right angle plus α. Thus, $\overline{C_{EF}C_{FG}C_{GH}C_{HE}}$ is a parallelogram by the equal segments on parallels theorem because $\overline{C_{EF}C_{HE}} = \overline{C_{EF}C_{FG}} = \overline{C_{GH}C_{FG}} = \overline{C_{GH}C_{HE}}$; specifically, it is a rhombus. By lemma 3.2.1, $\angle EC_{EF}F$ is right. $\overline{C_{EF}EC_{HE}} \cong \overline{C_{EF}FC_{FG}}$ holds the equality $\angle C_{HE}C_{EF}E = \angle C_{FG}C_{EF}F$. By addition, $\angle C_{HE}C_{EF}C_{FG}$ is right. By lemma 3.2.2, $\overline{C_{EF}C_{FG}C_{GH}C_{HE}}$ is a square. ■

No fair omitting Victor Thébault's name and using the word "rotate" to conceal all his hard work! Analogously, we have proven that the triangles of circumcenters and of incenters of the three triangles around the medial triangle are congruent. We did not say "translate" and call it done.

Later, we will define rotation in a way that does not involve motion and use it to solve some problems. But programing a computer to display a figure rotating is a demonstration, not a proof.

Advanced Orange–Belt Geometry

Sadly, some brilliant students will not go on to second-year *Geometry–Do*, either for financial reasons – they are in a private college preparatory academy and their parents cannot afford another year – or because they want to take calculus instead. I advise against this latter path.

It is not a good idea to test through calculus. I tested through the five-credit algebra class for beginning mathematics and engineering majors and instead took Calculus I, II and III all in my freshman year. This was a mistake because there were holes in my knowledge of advanced algebra that I would have filled had I taken that algebra class. Also, there are really annoying requirements (*e.g.,* government and world cultures) imposed on students of every major that are not going away; if you do not take them your freshman year, they will come back to haunt you your senior year. My advice for talented high-school students is to get as solid a grounding as possible in algebra and geometry, including advanced topics that are not normally taught in high school, but not to study calculus. As college freshmen, take advanced algebra and Euclidean geometry, along with the no-brainer classes like government and world cultures. This is an easy schedule of math you mostly already know, so there is time to study calculus on your own using one of the many available self-guiding textbooks. Take Calculus I, II, III and differential equations your sophomore year and ace them all. In your junior year, you can begin pursuing your specialty.

If not, then go your way, but I can at least leave you with some challenging orange-belt problems:

Problem 3.53 *Given square $\overline{EFGH}$, build equilateral triangles on $\overline{FG}$ and $\overline{GH}$, either both inside or both outside $\overline{EFGH}$, and with apexes J and K, respectively. Prove that $\overline{EJK}$ is equilateral.*

Problem 3.54 *Let $\overline{EF}$ be the diameter of a circle with center O and G be a point on the circle such that $\angle EOG < 2\varphi$. Let M be the intersection of the bisector of $\angle EOG$ with the circle. Let J and K be the intersections of the mediator of $\overline{OG}$ with the circle, with J on the M side. From O draw a line parallel to $\overleftrightarrow{MG}$ and let it intersect $\overline{FG}$ at I. Prove that I is the incenter of $\overline{FJK}$.*

Problem 3.55 *Given parallelogram $\overline{EFGH}$ and a circle centered at E tangent to $\overleftrightarrow{FH}$, let J be the intersection of it and $\overrightarrow{GE}$ extended past E. Construct a circle centered at G that is tangent to $\overleftrightarrow{FH}$ and let K be the intersection of it and $\overrightarrow{EG}$ extended past G. Prove that $\overline{JFKH}$ is a parallelogram.*

P. 3.53 is due to Thébault, a French geometer who would be appalled to hear Dieudonné shriek, "A bas Euclide! Mort aux triangles!" Why should hate for Debreu mean hate for Thébault?

Elementary Quadrature Theory

Theorems proving equality of areas are **quadrature** theory. This is blue belt because some results cite green- and red-belt theorems, and because quadrature introduces multiplication. In Eastern Europe, all high-school students must take three years of geometry, but decadent Western high schools require less, so we cannot assume that the general Western student will get to blue-belt study. Thus, we here prove some of the elementary results that cite only orange-belt theorems.

Until *Geometry with Multiplication*, we cannot calculate areas as we are yet unable to multiply the height and width of a rectangle. Does proving two areas equal without being able to calculate area seem fantastic? There is a physics analogy: $\mu = GM$ with μ the gravitational parameter for a body, G the universal gravitational constant and M the body's mass. For the Earth and the sun, scientists know μ quite accurately, as evidenced by their ability to calculate the flight of spaceships to the nearest meter, but they have only rough estimates of G and M. They are using the product without yet improving on Cavendish's 1798 torsion balance (it measures G) and being able to do the multiplication. We can equate areas because area is a magnitude and magnitudes are unique and an additive group. The union of some disjoint triangles has the same area as any others that have the same union. $\overline{EFGH} = \overline{EFG} \cup \overline{EGH} = \overline{EFH} \cup \overline{FGH}$ implies $\left|\overline{EFGH}\right| = \left|\overline{EFG}\right| + \left|\overline{EGH}\right| = \left|\overline{EFH}\right| + \left|\overline{FGH}\right|$. Minus, $-$, means removing a triangle. Replacing a union, $\overline{EFGH} = \overline{EFG} \cup \overline{EGH}$, with a sum, $\left|\overline{EFGH}\right| = \left|\overline{EFG}\right| + \left|\overline{EGH}\right|$, is left tacit.

Parallelograms and Triangles Area Theorem
All parallelograms with the same or congruent definitional triangles are of equal area.

> ***Proof***
> By choosing any two sides of a triangle, one as base and one as diagonal, it can be used in three ways to form a parallelogram. By definition of parallelograms, they are all double the area of the triangle. By transitivity, they are all of equal area. ■

Lemma 3.3 ***(Euclid, Book I, Prop. 35)***
Parallelograms with the same base and their opposite sides collinear are of equal area.

> ***Proof***
> Given parallelograms $\overline{EFGH}$ and $\overline{EFJK}$ with $\overline{GH} = \overline{JK}$ collinear, $\overline{HE} = \overline{GF}$ and $\overline{EK} = \overline{FJ}$. By adding or subtracting $\overline{KG}$ to/from $\overline{GH} = \overline{JK}$, we get $\overline{KH} = \overline{JG}$. By SSS, $\overline{KHE} \cong \overline{JGF}$. $\left|\overline{EFGH}\right| \pm \left|\overline{KHE}\right| \mp \left|\overline{JGF}\right| = \left|\overline{EFJK}\right|$, so $\left|\overline{EFGH}\right| = \left|\overline{EFJK}\right|$. ($\pm$ and $\mp$ means either $+$ and $-$, or $-$ and $+$; draw several figures with $\overline{GH} = \overline{JK}$ in different positions.) ■

Parallelogram Area Theorem ***(Euclid, Book I, Prop. 36)***
Parallelograms with equal collinear bases and their opposite sides collinear are of equal area.

Proof
Given parallelograms $\overline{EFGH}$ and $\overline{JKLM}$ with $\overline{EF} = \overline{JK}$ collinear; $\overline{GH}$ and $\overline{LM}$ collinear. Connect $\overline{EM}$ and $\overline{FL}$. By definition of parallelograms and transitivity, $\overline{EFLM}$ is a parallelogram. By lemma 3.3 and transitivity, $|\overline{EFGH}| = |\overline{EFLM}| = |\overline{JKLM}|$. ■

Let us pause to discuss Euclid's terminology. When Euclid says that figures are equal, he means that they are equal in area. This is also what I do except that I say "of equal area" to exclude other characteristics, like height. As I do, Euclid uses "equal" for angles and sides; *e.g., Prop. 6*, "If in a triangle two angles be equal, the sides which subtend the equal angles will also be equal." No modifier is needed because angles and sides have only one characteristic. For us, collinear is a set of points that are all on the same line. Euclid writes, "in the same parallels;" *e.g., Prop. 38*, "Triangles which are on equal bases and in the same parallels are equal." I write this, "Triangles with equal collinear bases and apexes on a line parallel to their bases are of equal area."

Triangle Area Theorem ***(Euclid, Book I, Prop. 38)***
Triangles with equal collinear bases and apexes on a line parallel to their bases are of equal area.

Proof
Given $\overline{EFG}$ and $\overline{JKL}$ with $\overline{EF} = \overline{JK}$ collinear and $\overleftrightarrow{GL}$ parallel to their line, find M and N on $\overleftrightarrow{GL}$ such that $\overline{EFGM}$ and $\overline{JKLN}$ are parallelograms. $|\overline{EFGM}| = |\overline{JKLN}|$ by the parallelogram area theorem. By definition of parallelograms, $\overline{EFGM}$ and $\overline{JKLN}$ are unions of congruent triangles so, if the parallelograms are of equal area, so are the triangles. ■

Euclid's contemporaries were astonished by the triangle area theorem because a triangle's perimeter can be increased without bound and yet retain the same area. A superficial analysis that compares triangles only when one is inside the other would conclude that perimeter and area go up and down together. It must be remembered that geometry was not a pastime for idle aristocrats, but was of intense interest to farmers, because their livelihoods depended on it. But they sometimes presented unsound arguments, like area and perimeter being directly related, which degenerated into fighting. Had Euclid not formalized deductive logic, so everyone could agree on what was proven, the farmers would have stabbed each other fighting over boundaries, nothing would have been planted, and people would have gone hungry. We are losing this certainty; the practice of propagandists to begin every sentence with "Statistics show that…" has left people in a fog of confusion with no means to distinguish between facts and alternative facts.

Triangle Area Theorem Corollaries (Euclid, Book I, Prop. 39, 40, 41)

1. *Triangles with equal collinear bases and apexes on lines parallel to and equidistant from the base line are of equal area.*
2. *Of triangles with equal and collinear bases on the same side of the base line, the locus of apexes such that the triangles are of a given area is a line parallel to the base line.*
3. *If a triangle has the same base as a parallelogram and its apex is on the parallelogram side opposite the base, or its extension, then the triangle's area is half the parallelogram's.*
4. *An orthodiagonal quadrilateral has half the area of the rectangle whose sides equal its diagonals.*

Triangle Area Theorem Converse

Triangles of equal area with collinear bases and apexes parallel to them have equal bases.

Proof

Given $\left|\overline{EFG}\right| = \left|\overline{E''F''G''}\right|$ with E, F, E'', F'' collinear and $\overleftrightarrow{GG''}$ parallel to this line, suppose the bases are unequal, $\overline{EF} > \overline{E''F''}$. Then there exists an M between E and F such that $\overline{EM} = \overline{E''F''}$. By the triangle area theorem, $\left|\overline{EMG}\right| = \left|\overline{E''F''G''}\right|$. But this is a contradiction because $\left|\overline{EFG}\right| = \left|\overline{EMG}\right| + \left|\overline{MFG}\right|$. Analogously if $\overline{EF} < \overline{E''F''}$. ■

Two Triangles Area Theorem

A median divides a triangle into two triangles of equal area.

Three Triangles Area Theorem

The three sides of a triangle as bases and the medial point as their apexes are of equal area.

Proof

Given $\overline{EFG}$ and medial point C, by the two triangles area theorem, $\left|\overline{GEM_{EF}}\right| = \left|\overline{GM_{EF}F}\right|$ and $\left|\overline{CEM_{EF}}\right| = \left|\overline{CM_{EF}F}\right|$. By subtraction, $\left|\overline{GEM_{EF}}\right| - \left|\overline{CEM_{EF}}\right| = \left|\overline{GM_{EF}F}\right| - \left|\overline{CFM_{EF}}\right|$, so $\left|\overline{GEC}\right| = \left|\overline{GCF}\right|$. Again, with another median and, by transitivity, all three are equal. ■

Six Triangles Area Theorem

The three medians divide a triangle into six triangles of equal area.

Medial Triangle Area Theorem

The medial triangle and the three triangles around it quarter the area of the parent triangle.

Medial Parallelogram Area Theorem I

The area of a medial parallelogram is half that of its parent quadrilateral.

Carpet Theorem I
Given square $\overline{EFGH}$, J an arbitrary point on $\overline{EF}$ and $K := \overline{EG} \cap \overline{JH}$, then $|\overline{EKH}| = |\overline{JKG}|$.

Heron has a blue-belt formula for triangle area given the sides: $A = \sqrt{s(s-e)(s-f)(s-g)}$.

Problem 3.56 ***(Euclid, Book I, Prop. 43)***
Given parallelogram $\overline{EFGH}$ and P on $\overline{FH}$, (1) Prove that $|\overline{EPH}| = |\overline{GPH}|$; (2) Draw lines through P parallel to the sides of $\overline{EFGH}$ and prove that the parallelograms with opposite vertices E, P and opposite vertices P, G are equal in area.

Problem 3.57 *Given $\overline{EFGH}$, draw a line through M_{FH} parallel to $\overleftrightarrow{EG}$ and let J be where it cuts $\overline{EF}$ (If it cuts $\overline{GH}$, then change the labels.) Prove that $\overline{GJ}$ bisects $\overline{EFGH}$; that is, $|\overline{JFG}| = |\overline{EJGH}|$.*

Lemma 3.4
The square on the leg of a right triangle is equal in area to the rectangle whose sides are the hypotenuse and the ***projection*** *of the leg on the hypotenuse.*

Proof
Given $\overline{EFG}$ with $\angle EFG$ right, construct external squares on each side; $\overline{EJKF}$, $\overline{FLMG}$ and $\overline{GNOE}$. Drop a perpendicular from F through F' on $\overline{GE}$ to F'' on $\overline{NO}$. Find P on $\overline{F'F''}$ such that $\overleftrightarrow{OP} \parallel \overleftrightarrow{EF}$. Find Q on $\overline{GF}$ such that $\overleftrightarrow{JQ} \parallel \overleftrightarrow{EG}$. By the parallelogram theorem, $\overline{EFPO}$ and $\overline{EJQG}$ are parallelograms. $\angle EFP = \angle QGE$ by the pairwise perpendiculars theorem, so $\overline{EFPO} \cong \overline{EJQG}$. $|\overline{EFPO}| = |\overline{EF'F''O}|$ and $|\overline{EJQG}| = |\overline{EJKF}|$ by the parallelogram area theorem. By transitivity, $|\overline{EF'F''O}| = |\overline{EJKF}|$. ■

Pythagorean Theorem ***(Euclid, Book I, Prop. 47)***
The square on the hypotenuse is equal in area to the sum of the squares on the legs.

Proof
By lemma 3.4, $|\overline{EF'F''O}| = |\overline{EJKF}|$ and $|\overline{GNF''F'}| = |\overline{FLMG}|$, using the same figure. $\overline{EGNO} = \overline{EF'F''O} \cup \overline{GNF''F'}$. Thus, $|\overline{EGNO}| = |\overline{EJKF}| + |\overline{FLMG}|$. ■

Problem 3.58
Prove that the squares on the diagonals of a parallelogram sum to the squares on the sides.

Lemma 3.5
Squares are congruent if and only if their sides are equal if and only if their areas are equal.

Pythagorean Theorem Converse ***(Euclid, Book I, Prop. 48)***
A triangle is right if the square on one side is equal in area to the sum of the other two squares.

Proof
Given $\overline{EFG}$ with the square built on $\overline{EG}$ equal in area to the sum of the squares built on $\overline{EF}$ and $\overline{FG}$, construct $\overline{JKL}$ such that $\overline{JK} = \overline{EF}$ and $\overline{KL} = \overline{FG}$ and $\angle JKL = \rho$. By the Pythagorean theorem and lemma 3.5, the squares built on $\overline{EG}$ and on $\overline{JL}$ are congruent, so $\overline{EG} = \overline{JL}$. By SSS, $\overline{EFG} \cong \overline{JKL}$, so $\angle EFG = \angle JKL = \rho$. ■

Diagonal Bisection Theorem
A diagonal divides a quadrilateral into two triangles of equal area iff it bisects the other diagonal.

Proof – citing only orange-belt theorems – is left as an exercise. It is foundational for blue belts. The following blue-belt theorems are in *Practical Shop Mathematics* by Wolfe and Phelps.

Projection Theorem (without proof)
The projection of a side of a triangle upon the base is equal to the square of this side plus the square of the base minus the square of the third side, divided by two times the base.

Intersecting Chords Theorem (without proof) ***(Euclid, Book III, Prop. 35)***
If two chords of a circle intersect inside the circle, the product of the two segments of one is equal to the product of the two segments of the other.

Intersecting Secants Theorem (without proof) ***(Euclid, Book III, Prop. 36, 37)***
If two secants of a circle intersect outside the circle, the product of the two segments of one, from the intersection to where the circle cuts it, is equal to the product of the two segments of the other, from the intersection to where the circle cuts it.

Altitude and Diameter Theorem (without proof)
The product of two sides of a triangle is equal to the product of the altitude to the third side and the diameter of the circumcircle.

Triangle Similarity Theorem (without proof) ***(Euclid, Book VI, Prop. 4, 5)***
If two triangles are similar, their corresponding sides are proportional.

Side–Splitter Theorem (without proof) ***(Euclid, Book VI, Prop. 2)***
A line through two sides of a triangle parallel to the third side divides those sides proportionally.

Problem 3.59 *Any line through two circles' touching point is cut in proportion to their diameters.*

Trigonometry has been eliminated and Geometry has been turned into a review of Algebra I – *Does everybody have the distance and midpoint formulas memorized yet?* – with three teachers, of Algebra I, II and Geometry, being told to toss in some trigonometry with their usual studies.

Suppose you employ three people and have a task with an easy part and a hard part. If you assign it to one employee, he will do it all – easy and hard – lest he be fired. If you tell all three to help complete the task, the easy part will be done three times, redundantly, and the hard part will not get done. Analogously, ladder-on-wall problems are taught three times; the laws of sines and cosines, never. *Common Core* is sometimes just plain wrong, but what it lacks most is leadership.

In the blue-belt chapter – sine and cosine are blue belt because they are ratios – geometry becomes more like algebra and the sides are given, or we are asked to find, numerical lengths in centimeters. $e = \overline{FG}, f = \overline{GE}$ and $g = \overline{EF}$ are abbreviations meant to aid in doing this algebra. The inscribed angle theorem is early in green belt; look it up so you can follow this next proof.

First Law of Sines $\frac{e}{\sin\alpha} = \frac{f}{\sin\beta} = \frac{g}{\sin\gamma} = 2R$

Proof

Given $\overline{EFG}$, in $\overline{M_{EF}OF}$, $\angle M_{EF}OF = \gamma$ by the inscribed angle theorem, $\overline{OF} = R$, and $\overline{M_{EF}F} = \frac{g}{2}$. Thus, $\frac{g}{2} = R\sin\gamma$. Rearranging, $\frac{g}{\sin\gamma} = 2R$. Analogously for $\frac{e}{\sin\alpha}$ and $\frac{f}{\sin\beta}$. ■

Second Law of Sines $\frac{e-f}{e+f} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$

From the first law of sines, $e = 2R\sin\alpha$ and $f = 2R\sin\beta$, which implies the second law of sines.

First Law of Cosines $g^2 = e^2 + f^2 - 2ef\cos\gamma$

Proof

Given $\overline{EFG}$ acute, let E' be the foot of the altitude from E to $\overline{FG}$. $\overline{EE'} = f\sin\gamma$ and $\overline{GE'} = f\cos\gamma$ and $\overline{FE'} = e - f\cos\gamma$ by the definition of sine and cosine in $\overline{EE'G}$.

$g^2 = (e - f\cos\gamma)^2 + (f\sin\gamma)^2$	Pythagorean theorem in $\overline{EE'F}$
$= e^2 - 2ef\cos\gamma + f^2\cos^2\gamma + f^2\sin^2\gamma$	Expand the squares
$= e^2 + f^2 - 2ef\cos\gamma$	$\cos^2\gamma + \sin^2\gamma = 1$ •

Proving this for when E' is on $\overrightarrow{FG}$ past G, or is on $\overrightarrow{GF}$ past F, is left as an exercise for the student. Also, verify that the largest interior angle of a $3 : 5 : 7$ triangle is $\frac{2}{3}\pi$. Good to know!

Second Law of Cosines $g = e\cos\beta + f\cos\alpha$

Prove this by substituting $\cos\alpha$ and $\cos\beta$ from the first law of cosines and simplifying to get g.

It is important that the vertex angles not be labeled A, B, C as most trigonometry textbooks do because A also means area. Use e, f, g for the sides because a, b, c are used in $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and, when using the quadratic formula to solve a triangle problem, it is confusing for a, b, c to have two meanings. Also, $a\sin(\quad)$ or $a\cos(\quad)$ look like $\operatorname{asin}(\quad)$ or $\operatorname{acos}(\quad)$, which is how we write these inverse functions because $\sin^{-1}(\quad)$ and $\cos^{-1}(\quad)$ look like $\frac{1}{\sin(\quad)}$ and $\frac{1}{\cos(\quad)}$.

Practical applications use tangent, but you usually do not want it when proving identities; convert it to sine over cosine. You *never* need cosecant, secant, or cotangent; these words only confuse. Terminology is the only thing that makes this difficult to prove: $\frac{1+\cot\theta}{\cot\theta} = \tan\theta + \csc^2\theta - \cot^2\theta$.

$\frac{1+\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}} \quad ? \quad \frac{\sin\theta}{\cos\theta} + \frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}$ Rid yourself of the confusing terminology!

$\frac{\sin\theta}{\cos\theta} + 1 = \frac{\sin\theta}{\cos\theta} + \frac{\sin^2\theta}{\sin^2\theta}$ $1 - \cos^2\theta = \sin^2\theta$ by the Pythagorean theorem.■

The Pythagorean equation, $\sin^2\theta + \cos^2\theta = 1$, is often cited, usually to simplify equations as above. It can convert an equation into all sines or all cosines, which allows you to factor it. Towards this end, $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ and $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$ and $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$ convert the arguments to θ. $\sin(\pi - \theta) = \sin\theta$ gives the two angles in ASS.

Angle Sum Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

These send $\cos(2\theta)$ to sine or cosine, to aid in factoring.

$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$ $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

It is easier to integrate a sum than a product; there is no need for the sum-product identities.

Surveying Techniques to Measure or Lay Off Lengths

Blue belts will learn how to find the area of an irregular quadrilateral when one side cannot be measured. Here we learn surveying fundamentals in preparation for these advanced techniques. The area of ranch land is the area of its projection onto the horizontal plane, not its actual surface area. If you are building a fence and need to know its length to estimate materials, install an odometer on a mountain bike that outputs to 0.01 km and put five magnets on the spokes instead of just one. Multiply by 200 to get meters; *e.g.,* if it says 5.73 km, it is really 1146 m. But this measurement will not do if you are buying the land and need to know its acreage. After some preliminary instruction, I will here explain how to accurately measure horizontal length.

It is often the case that existing fences appear to follow the path of a drunkard on his way home from the saloon. But the explanation is not (usually) drunkenness. It is because the fence builder was locating where to pound in fenceposts by sighting across the tops of two that he had already pounded in. There is much error in this procedure because, even with the aid of a level, none of the posts are perfectly vertical. If the fence is traversing a hill, all the errors will be in the same direction because the pounder tends to hammer his posts into the slope, so they tilt towards the downhill side of the fence. A fence that appears to be wandering aimlessly across the country is, on closer inspection, usually curving towards the downhill side of the slopes that it crosses.

Even without the purchase of expensive surveying equipment and the rather time-consuming need to re-level the tripod every time it is moved, making a fence straight can be achieved with a scoped rifle.[66] Hold the top of the vertical part of the crosshairs on a distant mountain peak and direct your helper to stick flags in the ground every five meters (or whatever the agreed upon distance between posts is) and in line with the bottom of the vertical part of the crosshairs.

Most hills become increasingly steep until they reach an inflection point, after which they become decreasingly steep and finally flat at the top. This inflection point is called the military crest of the hill because it is the highest point where a machine gun can be emplaced and cover all the hillside below it. The military crest is higher on the hill for a person standing upright than it is for a machine gun just centimeters off the ground.[67] Green belts will learn of machine gun emplacement and should know that, whatever other considerations must be taken into account, they should stay below the military crest of hills to avoid leaving blind spots at the base of the hill. Here, you need to know about the military crest because, when surveying for a fence that goes over hills, it is at the military crest of each hill where you position the telescope.

[66] Leave the rifle bolt behind so nobody accuses you of poaching cattle; that is a hangable offense in cattle country.
[67] Battles have been lost because an officer on horseback surveyed the battlefield and positioned his machine guns too far up the hill, which left a blind spot at the base of the hill where the enemy was protected from fire.

I will begin with an overview of how to build a fence. Cutting the bands on a new roll of wire can be dangerous because there is tension in the coils, and it can spring out at you. Tighten wire *only* with a hand-cranked pulley. *Never* use a tractor to tighten wire; if the wire snaps, it can disembowel you. If you do not wear prescription glasses, then wear dark glasses; when not under high tension, coiled wire tends to flip around unexpectedly and, if it scratches your eye, it can blind you. Leather gloves are a must, though they will not protect your hands entirely. Keep bandages and spray bottles of saline and of antiseptic in your truck; treat wounds immediately.

Disemboweled? If you are not now convinced that flipping hamburgers is the life for you, then let us look at how fences fail. Most fences are in hilly country; there are often gullies[68]. Fences fail because cows push their way through, but cows do so only if they see preexisting weakness.

1. **Untreated wooden fence posts rot.** Use only treated wooden posts, and only at the corners, gates, and bottoms of depressions. Use pounded-in steel posts between them.

2. **Cold weather tightens wire and pulls up posts.** Wire should be attached to the corner and gate posts but should otherwise slide in its fasteners. Do not over-tighten it.

3. **A corner or gate post topples over.** These posts should be wood; a second post is buried 1.3 meters away. On this post, the wire slides in its fastener just like on the steel posts. This post's purpose is to brace the corner post with a horizontal board and another board at an angle from the bottom of the brace post up to where the horizontal board meets the corner post. Cut slots in the posts for the horizontal board. Tighten a wire from the bottom of the corner post up to where the horizontal board meets the brace post.

4. **Posts at the bottom of a depression pull out.** Here you should have two wooden posts 2.4 m apart with a horizontal board and with wire at both angles (from the bottom of one to the horizontal of the other) and tightened enough to squeeze the board. If water flows in the depression, you may need boulders or solid concrete blocks to hold the fence down.

Three wires are not enough. Four is standard and five may be needed to enclose an alfalfa field, which cows lust after. Put the wire on the side that the cows are on; stand on the other side of the fence when tightening it. The actual mechanics of building fences cannot be explained here; work for someone who is experienced and observe him carefully, or take pictures of a well-made fence and try to duplicate that builder's work. If you come to a gully, put a corner post on each side and keep going. Someone else will have to solve that problem. The best solution is a culvert buried right up to the level, though this is expensive; dumping junk cars in the gully also works.

[68] Arroyos in Spanish, or wadis in Arabic; these terms are widely used even in English-speaking countries.

Find the endpoints of the fence. When the land was settled, surveyors may have buried markers, or there may be a fence that is known to have been professionally surveyed and that can be relied upon. A common situation is that you have only one endpoint or that you are constructing a rectangle on the side of a surveyed line. In America, ranch land is cut into 40-acre squares, which are 440 yards or 402.336 meters on a side. Louisiana may use French units and Texas may use Spanish units; there are other units in use around the world. 100 meters on a side is a hectare.

Measuring horizontal length requires a telescope with a bubble level, which is called a sight level. Also, you need a laser rangefinder; the ones sold in golf shops are better for surveying than the ones for hunters. They are more accurate than an odometer but can produce gross errors if the beam reflects off the wrong object; one should recon with a bicycle and recheck measurements that seem wrong. Observe the vertical height on a measuring rod. This measurement, h, is made more accurate if each man has a sight level and a measuring rod because they can steady their level by using their rod like a monopod and can average their measurements. r is the slant length and c is the correction; $c = r - x$. Then, $h^2 = r^2 - x^2 = (r - x)(r + x) = c(r + x)$ by the Pythagorean theorem, so $c = \frac{h^2}{r+x} \approx \frac{h^2}{2r}$. For gentle enough slopes that you can measure the vertical height on a rod, this approximation has negligible error. In hilly country, you must measure many short segments; on relatively flat land, you can measure a few longer segments.

The horizontal length is always less than the slant length; thus, in measurement work, you *subtract* the correction from the measured slant length to get the horizontal length. But, in layout work[69], you are given the horizontal length (*e.g.,* 440 yards if you are constructing a 40-acre plot of land) and you *add* the correction to this given length to know what slant length to look for.

Confusion about whether to add or subtract a correction is one of the most common causes of error, not just in surveying but in all branches of engineering. Thus, while modern laser rangefinders need no correction, I will here describe the old-fashioned use of a long steel tape. Steel has a coefficient of expansion of 0.000012; that is, it lengthens/shortens 0.000012 meters for every meter of its nominal length for each Celsius degree over/under 20℃.

Lay off 100 meters on a hot 37℃ day? Mark your point where the tape reads 99.98 meters.

Lay off 100 meters on a cold 3℃ day? Mark your point where the tape reads 100.02 meters.

On a hot 37℃ day, the measured length is 100 meters? The actual length is 100.02 meters.

On a cold 3℃ day, the measured length is 100 meters? The actual length is 99.98 meters.

[69] To lay off a length, extend a ray past a point by the given length out to a new point. Hence the term “layout work,” in contrast to “measurement work,” where you are given two points and want to know their separation.

If you position your telescope at the military crest of hills, you can usually move from hilltop to hilltop with no loss of accuracy. But what if you come to an obstacle that you cannot see through, and you are on flat ground so there is no elevated position that allows to see over it? Professional surveyors would use a transit – a telescope that can rotate both horizontally and vertically on a tripod – to measure four right angles. But a transit is a very expensive piece of equipment, it takes a long time to re-level it and orient its angle measurements with the compass each time it is moved, and it is difficult to read its double Vernier scales for degrees and for minutes of angle.

If F is the endpoint and E is a point such that $\overline{EF}$ is approximately the needed offset, construct an isosceles triangle $\overline{EFT}$ with the altitude from T about half $\overline{EF}$. Extend $\overrightarrow{ET}$ and $\overrightarrow{FT}$ that much again to G and H, respectively. By SAS, $\overline{EFT} \cong \overline{GHT}$, so, $\angle EFT = \angle GHT$, and, by the transversal lemma $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$. Extend $\overrightarrow{HG}$ past the house by at least $\overline{HG}$ and then construct another two triangles congruent to $\overline{EFT}$ as shown. Thales' diameter theorem (next chapter) states that $\angle EFG = \rho = \angle GHE$, so $\overline{FK} = \overline{GN}$, but the latter is measurable while the former is blocked.

Scenario: You passed the green-belt entrance exam! But there is trouble at home. With the birth of another baby brother, your house is crowded. If you cannot contribute some money so your parents can rent a bigger house, you will have to move out. You have heard of a farmer who purchased a run-down 40-acre farm; the fences have all fallen and cattle from an adjoining pasture have trampled the irrigation ditches. There is much work to be done! Wearing your green belt, you walk up to the farmer, shake his hand and say, "I am a *Geometry–Do* green belt. I know how to survey. I can build your fences arrow-straight and measure the area of your farm's projection onto the horizontal plane to confirm that it is the acreage you paid for."

"You're hired!" the farmer exclaims, amazed by his good fortune in meeting such a useful young man, "I'll pay you a hundred dollars a day, plus room and board." The corner posts of the old fence define the southern border, which is bracketed by right angles. The northern border is a straight river that runs at an angle to the southern border; the farm is a right triangle frustum.

With a 30-meter steel measuring tape, a hand level, and a meter rod, you take measurements. The slant length and the rise (declines are shown in red) are both measured in meters. The temperature is measured in degrees Celsius. Every pre-algebra student learns the formula for the area of a rectangle, $A = bh$, and for a triangle, $A = \frac{bh}{2}$, where b is base and h is height, which is, for a triangle, the apex altitude. Calculate hectares and then convert them to acres.

SW Corner to River

station	length	rise	temp.
1	30	0.05	20
2	30	0.02	22
3	30	0.01	23
4	30	0.04	24
5	30	0.01	26
6	30	0.15	27
7	30	0.18	28
8	30	0.23	28
9	30	0.19	29
10	30	0.26	30
11	30	0.31	32
12	30	0.29	32
13	30	0.34	31
14	30	0.27	17
15	30	0.35	19
16	30	0.31	20
17	30	0.35	22
18	11.31	0.27	23

SW Corner to SE Corner

station	length	rise	temp.
1	30	0.22	23
2	30	0.33	24
3	30	0.28	24
4	30	0.32	25
5	30	0.25	26
6	30	0.29	26
7	30	0.34	27
8	30	0.35	15
9	30	0.25	16
10	30	0.31	16
11	30	0.27	18
12	30	0.30	18
13	30	0.28	19
14	12.34	0.23	21

SE Corner to River

station	length	rise	temp.
1	28.34	0.03	22
2	30	0.07	23
3	17.83	0.06	23
4	30	0.26	25
5	30	0.32	09
6	30	0.38	12
7	30	0.39	12
8	30	0.43	13
9	30	0.45	15
10	27.19	0.42	16

The 28.34 m. and 17.83 m. lengths were to allow stretching the tape over gulleys. Objects you cannot see through, like a copse of trees, require constructing a rectangle as described on the previous page.

DO
INPUT $r_0,\ h_0,\ t$ ′ Measured slant length, rise and temperature.
$n = n + 1$ ′ To conclude, input r_0 longer than your tape, 30.
$c = 0.000012 r_0 (t - 20)$
IF $r_0 = 30$ THEN $r = r_0 - c$ ELSE $r = r_0 + c$
$h = h_0\big(1 + 0.000012(t - 20)\big)$
$x(n) = \sqrt{r^2 - h^2}$ ′ $x(n) \approx r - \frac{h^2}{2r}$ is used when $\sqrt{\ }$ is unavailable.
LOOP UNTIL $r_0 > 30$
FOR $i = 1$ TO $n - 1$: $x = x + x(i)$: NEXT i : PRINT x ′ The answer is $x = 39.995$ acres.

Taking slope and temperature into consideration is what divides professional surveyors from amateur surveyors. I grew up on a cattle ranch 2500 meters above sea level. Much of my work every summer was fixing fences; many staples had pulled out of the wooden fence posts. Why? Because the $-40°$ C. winter temperatures caused the wire to contract enough to pull on the posts. So don't tell me temperature has no effect! Also, even as a boy, I wondered at the perfect squares on the surveyor's map. This made sense only if the map was a projection on the plane.

Basic Terminology Used in Surveying

You cannot be an official surveyor without a license but, if you get a job as a surveyor's helper, knowing the lingo and showing an understanding of what your boss is doing will help you get promoted. He has no time to answer basic questions but, if you already know the basics, he may help you study for the licensing exam. If nothing else, it gets you out of a trench shoveling.

The beginner section, *Defense Positioning and Geometry*, makes it clear that machine gunners want a smooth slope that they can graze with their fire; enemy infantry can get behind the high points and into the low points. A smooth slope is also the desire of farmers, so the high points are not left bare, and the low points do not become a swamp. With slight modification, the techniques described in the previous section can be used for **giving grade**, or **grade staking** as it is sometimes called. This helps the excavators remove dirt from the **cuts** and put it in the **fills**. Boldface indicates surveyor terms; they are not in the geometers' glossary.

In the previous section, I described using a scoped rifle to build a fence straight. A **transit** is a telescope that can rotate both horizontally and vertically on a tripod. (A **theodolite** is the same thing, but it is more accurate.) It is expensive and the need to re-level it every time the tripod is moved to a new **station** is time-consuming and tedious. But its advantage over the rifle is that it has a built-in compass and that it can be rotated precisely $180°$ to take a **backsight** on the fence that has already been built, while the rifle can only be used for **foresight**. A **level** is like a transit, but it does not tilt, and it does not measure angles, though it does have notches at $90°$ and $180°$ so it can be used to take a backsight or to construct a right angle. The tripod is at eye level so, when climbing a hill, the next station cannot be so far that the **elevation** has increased more than 1.5 meters. Stations can be farther apart when descending a hill because the **rod** is longer than 3 meters. To **balance foresights and backsights** means that, while the stations do not have to be exactly the same distance apart, as they were in the example with the 30-meter **tape**, they should be roughly so; if they are clumped together on the uphills and spread out on the downhills, the measurement is different than it is for another surveyor moving in the opposite direction.

The quadrature chapter will assume only a laser rangefinder, not a transit, because angle measurements lend themselves to trigonometry. There is nothing wrong with trigonometry – it is an interesting subject too – but it is not what this book is about. The orange-belt theorem that is most relevant to surveying is the quadrilateral angle sum theorem. If you measure the angles at each of the four vertices, they should add up to $360°$, which is called **closure**. This gives your results far more credibility than a series of roughly collinear stations; a mistake at any one of them renders the rest meaningless. Transits are error-prone devices; mistakes happen. **Closing the horizon** means adding up the angles around a point to get $360°$, another check on accuracy.

Military surveyors are engaged in laying out static defenses with guns in concrete bunkers. The guns will be roughly collinear, but the surveyor should still obtain closure – using the polygon angle sum theorem – by looping around through stations at markers behind the line. These markers can then serve as reference points when guns are added or repositioned.

There are also reference points in the enemy's territory, which are prominent landmarks that are visible even in the smoke and dust of a battle. Since you cannot go to them to set up your transit – the enemy is kind of fussy about that – the surveyor must **triangulate** their position. This means that he uses the ASA theorem to construct a fully defined triangle with a known base and the enemy reference point at the apex. Generally, there is a hill in the interior of the triangle so your mortar battery, located on the base of the triangle, does not have line of sight on the apex and, most importantly, enemy gunners do not have line of sight on them.

Speed is of the essence. You do not have ten years to be built static defenses, like the French had for their Maginot line; it is entirely possible that the enemy will attack while you are still doing calculations. This is where geometry shines compared to trigonometry, though computer software has recently been introduced that can calculate trigonometry faster than one can draw a geometric figure. But, if your tools consist of a map, a compass, a ruler, a protractor, and a scientific calculator, you will be wise to set the calculator aside and just draw the figure. The upside of the scientific calculator is that it provides you with twelve decimal digits of accuracy; the downside is that you are dead before you have completed the calculation. Drawing the figure on a map and measuring angles with a protractor and distances with a ruler only provides three decimal digits of accuracy, but it puts mortar rounds at least *near* the enemy before they kill you.

An **azimuth angle** is measured clockwise from magnetic north. This differs from the trigonometrician's practice of measuring angles counterclockwise from due east. It also differs from a mariner's **bearing**, which measures the difference from north or south towards east or west; *e.g.,* bearing $S22°E$ is azimuth $158°$. The **angle of elevation** is on the vertical plane from $-90°$ (down) to $90°$ (up). In gunnery, you can ignore declination because all angles are relative.

Problem 3.60

From your mortar, M, you extend a line 170 meters with an azimuth angle of $107°$ to F. Backsight and extend 170 meters to E. If G is an enemy gun to the north, $\angle EFG = 67°$ and $\angle FEG = 76°$, at what azimuth angle and range should the mortar gunner be instructed to fire his weapon? (It is best if maps are scaled so 1 cm is 10 m. Here, 5 mm equals 10 m fits on U.S. letter-size paper.)

Solution

From M draw a ray to north and then $\overleftrightarrow{EF}$ at $107°$. Construct the triangle by ASA. Measure the azimuth angle from M to G at $12°$; measure the range at 506.5 meters. ■

How Military Surveying Differs from Civilian Surveying

First, we will consider two problems that seem to have nothing to do with surveying but are analogous to two surveying problems. Suppose that Musclehead Mike and Beanpole Bill are employed as ditch diggers. Mike can dig a 10-meter ditch in 5 hours while Bill can dig a 10-meter ditch in 8 hours. But their boss is in a hurry, so he has Mike start on one end and Bill start on the other end. How long will it take for Mike and Bill, working together, to dig a 10-meter ditch?

Many students stumble over this problem because they are used to thinking of rates as being units of distance divided by units of time, like $\frac{m}{h}$. But the ditches being dug are always the same length, so the 10-meter figure is just thrown in to confuse students – we could have said "standard ditch" without specifying the length in meters. The rates are $5 = \frac{t_M}{n}$ and $8 = \frac{t_B}{n}$ where t_M is the time it takes Mike to dig n ditches and t_B is the time it takes Bill to dig n ditches.

Solve both equations for n and then add them together to get the number of ditches dug by Mike in t_M hours and Bill in t_B hours: $\frac{t_M}{5} + \frac{t_B}{8} = n$. But $t_M = t_B$ in this case because both boys finish simultaneously when they meet, so we will call this time t. And $n = 1$ because they have only one ditch to dig. Thus, for this problem, $\frac{t}{5} + \frac{t}{8} = 1$. Getting a common denominator yields $\frac{8t+5t}{8\times5} = 1$, so $t = \frac{8\times5}{8+5} = \frac{40}{13} \approx 3.08$ hours. It is the infamous product-over-sum formula!

Now let us solve a more general problem of several – we will say three – people working together to accomplish n tasks. Suppose their rates are r_1, r_2, r_3 hours per task. $t = \frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_1 r_3} n$. This problem is often posed as several pumps filling (or draining) n water tanks, with the strength of the pumps being given in hours per tank. This is only true if the people can work independently.

The product-over-difference formula is derived in almost the same way and is also characterized by the rates being given in time per task. Suppose that Mike and Bill are running laps in P.E. class. Bill runs at a steady 70 seconds per lap while it takes Mike 100 seconds to haul his big butt around the track. How long will it take for Bill to lap Mike? $\frac{t}{100} + 1 = \frac{t}{70}$. Rearranging, $\frac{t}{70} - \frac{t}{100} = 1$. Getting a common denominator yields $\frac{100t-70t}{100\times70} = 1$, so $t = \frac{100\times70}{100-70} = 233.\overline{33}$ seconds. If this is a 1600-meter race and the track is a 400-meter oval, will Bill lap Mike? Yes, in $\frac{233.\overline{33}}{70} = 3\frac{1}{3}$ laps.

Suppose that two surveyors want to measure the altitude of a blimp tethered over their city. They could get directly under the blimp and aim their laser rangefinder upwards. If there are obstructions, they can aim it from two points and then find the altitude of this triangle by using problem 1.25. Suppose they are collinear with the projection and 300 meters apart. The nearer one measures a diagonal distance of 513 meters and the farther one measures a diagonal distance of 750.5 meters. If x is the distance of the farther one from the projection, then, by the Pythagorean theorem, $750.5^2 - x^2 = 513^2 - (x - 300)^2$. Solve for $x = 650$ meters and then $h = \sqrt{750.5^2 - 650^2} \approx 375$ meters. But, if the blimp is tethered over their city because an occupying army has surveillance cameras, snipers, and air-to-ground missiles on it, then aiming a laser rangefinder at the blimp is dangerous. They can see your laser beam and will fire on you.

Suppose the surveyors position themselves collinear with the blimp on opposite sides of it and 1000 meters apart. They use transits – devices that can measure the **angle of elevation** to a target – to measure angles of $30°$ and $47°$. We do not know how far either of them are from the projection of the blimp, so we call the distance of the farther surveyor from the projection x meters and the distance of the nearer surveyor from the projection $1000 - x$ meters. If h is the unknown height of the blimp, then $\tan 30° = \frac{h}{x}$ and $\tan 47° = \frac{h}{1000-x}$. By reasoning analogous to the ditch-digging problem above, $h = \frac{\tan 47° \times \tan 30°}{\tan 47° + \tan 30°} 1000 \approx 375$ meters.

Suppose the surveyors position themselves collinear with the blimp on the same side of it and 300 meters apart. They use transits to measure angles of $30°$ and $47°$. We do not know how far either of them are from the projection of the blimp, so we call the distance of the farther surveyor from the projection x meters and the distance of the nearer surveyor from the projection $x - 300$ meters. If h is the unknown height of the blimp, then $\tan 30° = \frac{h}{x}$ and $\tan 47° = \frac{h}{x-300}$. By reasoning analogous to the foot-race problem above, $h = \frac{\tan 47° \times \tan 30°}{\tan 47° - \tan 30°} 300 \approx 375$ meters.

If the surveyors are not collinear with the projection, after measuring the angle of elevation, they must lower their transits to the level and measure the transversal angle to the other surveyor. If the nearer surveyor measures a $19.0°$ angle and the farther surveyor measures a $10.1°$ angle, by the angle sum theorem, the vertex at the projection has a $150.9°$ angle. If they are 971 meters apart, $\frac{971}{\sin 150.9°} = \frac{x}{\sin 10.1°}$ by the Law of Sines, with x the distance from the nearer surveyor to the projection. $x = 350$ meters. Thus, $h = 350 \tan 47° \approx 375$ meters. Beware! Had we known $x = 350$ meters, $\operatorname{asin} 0.4865 \approx 29°$. There are two possible angles, and you get the acute one!

In civilian use, surveyors aim their rangefinder at a reflector held by their rod man. Most lasers cannot measure 971 meters when aimed at a man wearing unreflective cotton clothes. So, each aims at a metal light pole and measures the angle between it and the other surveyor. If one measures 473 meters and $11°$ and the other measures 518 meters and $12°$, then, by the Second Law of Cosines, the distance between them is $473 \cos 12° + 518 \cos 11° \approx 971$ meters.

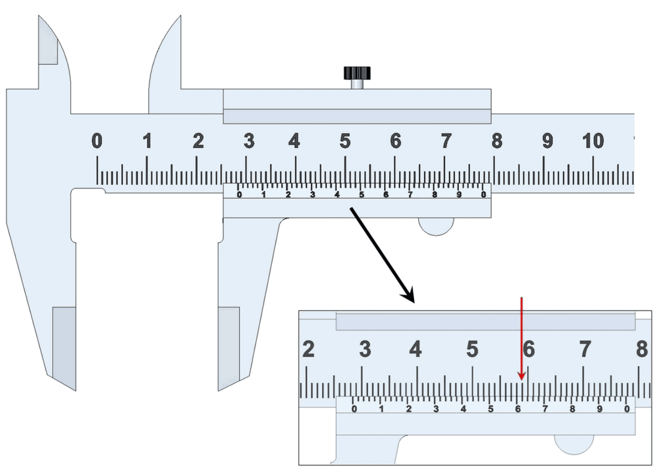
How to Apply for a Job that Uses Geometry

In most of the world (*e.g.,* India and Russia), graduating from middle school requires three years of geometry, and aspiring mathematicians and engineers get another two years in high school. But America requires only one year of geometry so, sadly, there are some of you whom I will not be seeing again. But I can at least give you some pointers on leveraging the little bit of geometry that you do know into a job of the type that requires being interviewed by a practicing engineer.

1. The interviewer is an engineer, but that is not the job you are applying for. So, don't tell him about your ability to prove specific geometry theorems – he already knows them – but speak in general terms about how the study of geometry has taught you to employ cool logic in the face of adversity. He has known adversity and he has seen people panic.

2. He will take you on a tour of the building. For older applicants with experience, this tour is meant to demonstrate that his is a state-of-the-art facility and to show them the machine that their résumé claims that they have experience on to see if they really do. But he knows that you have no experience, so why do you suppose he is taking you on this tour? He is watching to see that you walk briskly and with your head up. If you shuffle like an old man or swagger like a thug, he will not hire you. Never, *never* lean on a wall.

3. He will hand you a Vernier caliper or micrometer and ask you what it reads. If you stumble over this simple task, he will not hire you. The standard unit of measurement is the millimeter; the illustration shows 28.62 mm. In America, the standard unit is the mil, which is 0.001 in. Even if the last digit is zero, you report it in mils; *e.g.,* 0.23 inches is 230 mils. American calipers are like the one shown, but the units are inches, and the Vernier scale is subdivided into ten parts, not five. A micrometer is analogous, but the scale is on a cylinder with a turn knob. This is important! Buy one and practice at home.



Squares and Rectangles and Rhombi! Oh My!

Past *Regents* Examinations in Geometry (*Common Core*)[70] are available on the internet. Most of the questions require a knowledge of squares, rectangles, and rhombi. *Regents* does not require you to prove them, only to memorize them. Indeed, for most problems you do not need to have any theorems memorized because the exam shows a figure and asks which of four statements is *not* necessarily true. They all *look* true but, if you redraw the figure using the given information but with different lengths and angles, it will be obvious which statement no longer looks true.

Once I caught on to the test-taking technique of redrawing figures with different lengths and angles and then observing which of the four possible answers no longer looks true, I realized that it is entirely possible to pass this exam – though not to ace it – without having ever opened a geometry book. Most of the questions are meant to catch out students who base their answers entirely on the appearance of the given figure. Apparently, it did not occur to the examiners that students could redraw the figure themselves.[71] Plugging in the five answers also works.[72]

Common Core theorems come in two groups: squares, rectangles, and rhombi, the subject of this section; and similarity, the work of blue belts. But almost all the similar triangles are nested or crossed – the easiest kind. All that is really needed to solve these problems is knowing how to cross multiply. Cross multiplication is an important skill! In fact, *Regents* seem to feel that any problem involving cross multiplication is a geometry problem, even in the absence of triangles.

> *Seawater contains approximately 1.2 ounces of salt per liter on average. How many gallons of seawater would contain 1 pound of salt?* (Problem #18, June 2016)

Cross multiply to get the answer in liters; panic time if you are unaware that there is a chart on the back page with the liters-to-gallons conversion and other factoids that have nothing to do with geometry. That back-page chart is important! Many of the problems require students to plug numbers into formulas, so you will want to tear it out of your exam booklet and place it prominently on your desk, so no errors are made transferring formulas while flipping through the booklet. Two terms that do not appear on the back page are rectangular prism – a fancy name for a box – and sector, which is like a slice of pizza; its central angle is to 360° as its area is to πr^2.

Many of the problems require basic trigonometry (no identities; just the definitions of sine, cosine, and tangent) and basic algebra. Ladders leaning against walls is a popular topic. If a

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[70] An algebra exam masquerading as a geometry exam: www.nysedregents.org/geometrycc

[71] The GRE asks us to compare $\angle EGM_{EF}$ to $\angle F$. They illustrate with a nearly equilateral triangle, so $\angle EGM_{EF} < \angle F$. But, if you draw a triangle with a very small $\angle F$ and an obtuse $\angle G$, it is clear that the answer is indeterminate.

[72] What the SAT teaches: blog.prepscholar.com/plugging-in-answers-a-critical-sat-math-act-math-strategy

problem gives the measure of the angle of elevation to a point from two points on the ground, use the definition of tangent to set up two linear equations and then solve them simultaneously. Parabolas and ellipses are not required, but circles are; know how to complete the square to get $r^2 = (x - x_0)^2 + (y - y_0)^2$ and the meaning of x_0, y_0, r. Given two points, be able to find the line through them, the distance between them, and their midpoint. Also, find a line through a point given its slope or given a perpendicular line. *Common Core* "geometry" is almost all algebra and there is no new algebra to learn – this is all from freshman Algebra I. There is trigonometry, but only the definitions of sine, cosine, and tangent; *Common Core* never asks for familiarity with or proof of trigonometric identities. With proportions, algebra is $\frac{3}{4}$ of the exam – enough to pass!

Squares, Rectangles and Rhombi Theorem

1. *The diagonals of a rhombus bisect each other and the vertex angles.*
2. *The diagonals of a rhombus are perpendicular. (The converse is not necessarily true.)*
3. *The diagonals of a rectangle are equal. (The converse is not necessarily true.)*
4. *A parallelogram is a rectangle if and only if its diagonals are equal.*
5. *A parallelogram is a rhombus if and only if its diagonals bisect the vertex angles.*
6. *In an isosceles triangle frustum: (1) base angles are equal; (2) opposite angles are supplementary; (3) legs are equal; and (4) diagonals are equal. And the converses.*
7. *The area of a square is half the area of the square built on the diagonal.*

All seven of these statements are easy and are left as exercises. It is, frankly, incredible that this is almost all that an American high-school student needs to know about geometry to graduate and to get a high enough score on the SAT to be accepted into college. (Hint! All that American colleges really want is your money. If you are a moron, then that is just all the more money for them when you have to pay for remedial college math classes.) And the *Common Core* student does not even need to know how to prove them! Memorization is all that is required.

Construction 3.19 ***(Euclid, Book IV, Prop. 11)***

Inscribe a regular (equilateral and equiangular) pentagon in a circle.

Solution (without proof)

By C. 3.1, locate the circle center, O, if it is not already known. Draw a diameter $\overline{EF}$ and, by C. 1.3, raise a perpendicular from O to intersect the circle at G. Let the intersection of an arc with center M_{EO} and radius $\overline{M_{EO}G}$ intersect $\overline{OF}$ at P. $\overline{GP}$ is the length of a side. ■

Proof waits for blue belts to prove the mean ratio theorem; but, for now, just memorize the construction. If you plan to sell your soul to Satan, you will need to draw this on the floor.[73]

[73] Did you hear about the dyslexic devil worshipper? He sold his soul to Santa! 😊

The inscribed octagon theorem was in the orange-belt exit exam. Did you get it?

Inscribed Octagon Theorem
Given a square with circles around each vertex of radii equal to half the diagonal, the circles cut the square at the vertices of a regular octagon.

> ***Proof***
> Given $\overline{EFGH}$ square with center O, lay off $\overline{EO}$ on $\overrightarrow{EF}$ and $\overrightarrow{FE}$ to J and K, respectively. $\overline{EO} = \overline{FO}$ and $\angle OEJ = \angle OFK = \frac{\rho}{2}$ by the squares, rectangles, and rhombi theorem #1; ρ is a right angle. By SAS, $\overline{OEJ} \cong \overline{OFK}$. By the isosceles angle theorem, their base angles are $\angle OKJ = \angle OJK = \frac{3}{4}\rho$. By the isosceles triangle theorem converse, $\overline{KJO}$ is isosceles; by the angle sum theorem, its apex angle is $\angle JOK = \frac{\rho}{2}$. By the isosceles angle theorem, the supplements of the base angles of $\overline{OEJ}$ and $\overline{OFK}$ are $\angle EKO = \angle FJO = \frac{5}{4}\rho$. By AAS, $\overline{EKO} \cong \overline{FJO}$ and, by the angle sum theorem, their apex angles are $\angle KOE = \angle JOF = \frac{1}{4}\rho$.
>
> Lay off $\overline{EO}$ on $\overrightarrow{GF}$ to L. By an analogous construction, $\angle LOF = \frac{1}{4}\rho$ and $\overline{LO} = \overline{JO}$. By SAS, $\overline{JOK} \cong \overline{LOJ}$. Analogously, there are eight congruent triangles; thus, an octagon. ■

Suppose that you are tasked with this in a beginning algebra class. Unsure of how to begin, you draw four segments cutting out the corners at 45° angles and ask yourself, what must be true of these cuts for the figure to be a regular octagon? The angled segments must be equal to the uncut sections of the sides, between the cuts. And what are these lengths? If this is a unit square and what is cut from each side is of length x, then the uncut section is of length $1 - 2x$. The hypotenuse of the cut-out right isosceles triangle is $\sqrt{2}x$. Let us set them equal!

$\sqrt{2}x = 1 - 2x$	What must be true for the octagon to be regular
$2x^2 - 4x + 1 = 0$	Square both sides and collect like terms
$x = 1 - \frac{\sqrt{2}}{2} \approx 0.2929$	Quadratic formula; the other solution is too long

We could stop here, draw a 10 cm square and then use calipers to lay off 2.93 cm, but it would be better if we could relate this to the square somehow. x and $1 - 2x$ are collinear, so let us add them together: $\frac{\sqrt{2}}{2}$. Now we are cooking with gas! By a lucky insight, we recognize this as a number we have seen before. *But where???* Just by randomly scanning our eyes over the figure, we spot it: $\frac{\sqrt{2}}{2}$ is half of the square's diagonal. Even with no explanation for why these segments are equal, we then claim to have proven the inscribed octagon theorem. *What a lucky insight!!!*

Algebraists boast of this "elegant" construction as an example of their prowess in "analysis," but they nowhere establish cause and effect between the two lengths. They just happened to notice that one segment in their figure is labeled $\frac{\sqrt{2}}{2}$ and another segment has the same $\frac{\sqrt{2}}{2}$ label. My, what a happy and unexpected coincidence! This is considered a proof in algebra because $\frac{\sqrt{2}}{2}$ is the same length no matter where it appears. But, in geometry, we demand an explanation of cause and effect. The student cannot measure a length with his compass, randomly drop his compass around the figure until he finds another segment of the same length, and then shout *"Eureka!"* There are elegant proofs in algebra (*e.g.,* $\sqrt{2}$ being irrational), but this is not one of them. *Common Core* shills like this proof because they have never studied geometry and they need an excuse to replace geometry with algebra. But the role that such happy coincidences play in algebra excludes algebra from instructing students in logic, which is our purpose.

The next problem is also one that algebraists boast of, though with even less justification. Note that it is also true for a parallelogram, $\overline{EFGH}$, but proving it for parallelograms is black belt.

Lemma 3.6

Let ρ be a right angle, σ be a straight angle and φ be the interior angle of an equilateral triangle. φ trisects σ and $\frac{1}{2}\varphi$ trisects ρ. The exterior angle of an equilateral triangle is $\rho + \frac{1}{2}\varphi$.

Dakota Defense Problem

Given a rectangle, $\overline{EFGH}$, find J on $\overleftrightarrow{FG}$ and K on $\overleftrightarrow{GH}$ such that $\overline{EJK}$ is an equilateral triangle.

Solution

Build an equilateral triangle on $\overline{GH}$ with its apex, P, on the same side of $\overleftrightarrow{GH}$ as E and F. Let $J := \overrightarrow{EP} \cap \overleftrightarrow{FG}$. Build an equilateral triangle on $\overline{FG}$ with its apex, Q, on the same side of $\overleftrightarrow{FG}$ as E and H. Let $K := \overrightarrow{EQ} \cap \overleftrightarrow{GH}$. It is proven below that $\overline{EJK}$ is equilateral.

Proof

Assume that $\overline{FG} \leq \overline{GH}$; if it is not, relabel. By the center line theorem, P is on the mediator of $\overline{GH}$. By the transversal theorem corollary and the triangle frustum mid-segment theorem converse, $P \equiv M_{EJ}$. By the center line theorem and the mid-segment theorem, Q is on $\overleftrightarrow{M_{FG}M_{FK}}$; by the mid-segment theorem, $Q \equiv M_{EK}$. $\overline{EF} = \overline{HG} = \overline{PG}$; and, by lemma 3.6, $\angle EFQ = \angle PGQ = \frac{1}{2}\varphi$; also, $\overline{FQ} = \overline{GQ}$. Thus, by SAS, $\overline{EFQ} \cong \overline{PGQ}$, which holds the equalities $\angle FQE = \angle GQP$ and $\overline{QE} = \overline{QP}$. If the angle between them, $\angle PQE$, is φ, then $\overline{PQE}$ is equilateral. $\angle PQE = \angle GQF + \angle FQE - \angle GQP = \angle GQF = \varphi$. By medial triangle theorem I, $\overline{PQE}$ equilateral implies that $\overline{EJK}$ is equilateral. ■

The Dakota Defense is of interest to military cadets. Soldiers are sometimes tasked with building something that they know will be targeted by the enemy – say, a munitions dump – in the middle of open farmland that has been cut into rectangles by paved roads. They know:

1. Their bases must be on paved roads, so they can quickly move to confront enemy infantry approaching from anywhere, and so they can enfilade the roads to hit enemy vehicles.

2. Enemy aircraft are best met by anti-aircraft guns at the vertices of an equilateral triangle.

Helping the U.S. military fight more effectively is also why *Geometry–Do* will emphasize machine gun emplacement in the green-belt chapter. Russia teaches real geometry in their high schools, not that bogus *Common Core* drivel, and we must too if we are going to fight them. It is ridiculous that American military officers go into battle without a scientific approach to laying ambushes.

But, right now, let us get back to the task at hand: mocking algebraists who are always on the lookout for an excuse to replace geometry with algebra. Paul Yiu[74] writes:

> *This construction did not come from a lucky insight. It was found by an analysis! Let* $\overline{EF} = \overline{GH} = a, \overline{FG} = \overline{EH} = b$. *If* $\overline{FJ} = y, \overline{HK} = x$ *and* $\overline{EJK}$ *is equilateral, then a calculation shows that* $x = 2a - \sqrt{3}b$ *and* $y = 2b - \sqrt{3}a$. *From these expressions of* x *and* y*, the above construction was devised.*

Isn't that amazing? Paul Yiu "devised" a construction based entirely on two algebra equations!

> *D. E. Smith (2013, p. 95) explained that the teaching of constructions using ruler and compass serves several purposes: "it excites [students'] interest, it guards against slovenly figures that so often lead them to erroneous conclusions, it has genuine value for the future artisan, and it shows that geometry is something besides mere theory…" For all the strength and power of algebraic analysis, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for mere constructability.*

After quoting D. E. Smith on ruler and compass constructions, Paul Yiu then denounces them as "impractical," settles for "mere constructability," and extols the "strength and power" of algebra. Absurd! Yiu did not prove anything; he just smeared some algebra over someone else's result. The geometer that he stole from drew an almost square rectangle; Yiu does not know how to do this for skinny ones. *Forum Geometricorum* is a charade; it is algebra masquerading as geometry.

[74] Yiu, Paul. 2005. "Elegant Geometric Constructions." ***Forum Geometricorum***. 5: 75-96 math.fau.edu/yiu/GeometricConstructions/ElegantGeometricConstructions.pdf

A Brief Introduction to Linear Algebra

Having mocked algebraists who are always on the lookout for an excuse to replace geometry with algebra, we must now confront the reality that passing standardized geometry exams in America requires a rudimentary knowledge of linear algebra. Unfortunately, Jason Zimba had his head up his you-know-what when he defined the high-school mathematics curriculum, because matrices are in Algebra II, which comes after Geometry. *Oops!* Thus, for the remainder of this chapter, we will diverge from geometry to help students with Zimba's annoying fill-in-the-bubble exams.

$\begin{matrix} ex + fy = u \\ gx + hy = v \end{matrix}$ $\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ These are equivalent expressions of two linear equations in two unknowns, x and y, given constants e, f, g, h, u, v.

Solving the first equation for y yields $y = -\frac{e}{f}x + \frac{u}{f}$, which is the slope-intercept form of a line, so it should be clear that the two equations on the left are two lines, which are either concurrent, parallel, or intersect at a point. But the student may not be familiar with the matrix form of this. The square matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ is simply four numbers arranged in a square pattern. $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ are two numbers arranged one above the other. Rectangular arrangements and numbers side by side are also possible, but the two arrangements shown are all that is needed now. Placing two matrices together, as $\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, represents multiplication. Matrix multiplication is not a symmetric relation. Switching the order of the two multiplicands generally does not have the same product and, indeed, the multiplication may not be defined both ways. The beginning student does not need a thorough understanding of matrix multiplication; he only needs to know that $\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ is defined and it means $ex + fy = u$ and $gx + hy = v$, simultaneously.

The student should know that this expression is sometimes abbreviated $\left[\begin{array}{cc|c} e & f & u \\ g & h & v \end{array}\right]$; it is simply stated in the text that x and y are the two unknown variables. x and y are used throughout pure math; but, in applications, one must state what variables are being solved for. This format is helpful when solving the equations using Gaussian elimination, which will not be explained here because it is mostly of interest to computer programmers. Instead, we will explain how to solve systems of linear equations using Cramer's rule, which requires this format: $\left[\begin{array}{c|cc|c} u & e & f & u \\ v & g & h & v \end{array}\right]$.

Linear systems of equations are used in high-school physics for the motion of projectiles, billiard balls, and the analysis of electrical circuits. Because it is so quick and efficient, Cramer's rule is what separates A students from B students; the latter run out of time on exams. Sadly, Algebra II comes after Physics. The formerly straight-A student can thank Jason Zimba for his B in physics.

Cramer's rule is delayed because students find matrix arithmetic complicated. In America, geometry is a sophomore class that comes between Algebra I and II. *Common Core* turns triangle congruence into a review of Algebra I; the congruent triangles are just an excuse for attaching linear equations to two lengths, two angles or – *God forbid!* – a length and an angle.[75] This is wrong on so many levels! (1) It ignores real geometry; (2) It is bad Algebra I to add lengths and angles; and (3) Cramer's rule is in Algebra II, so it is taught after it is needed.

Glencoe Geometry (p. 256) declares two triangles congruent, one with all its sides and angles labeled: $a = 38.4$ mm, $b = 54$ mm, $c = 32.1$ mm and $\alpha = 45°$, $\beta = 99°$, $\gamma = 36°$. The other triangle has the side corresponding to a labeled $(x + 2y)$ mm and the angle corresponding to β labeled $(8y - 5)°$. *Glencoe* then solves $8y - 5 = 99$ and $x + 2y = 38.4$ simultaneously to get $x = 12.4$ and $y = 13$. The former equation implies that y is an angle and the latter equation implies that x is something that, when added to an angle, is a length. *What?* x is the mysterious lengle! This is stupid – *Look at the units!* – but at least we can solve the equations efficiently.

$\begin{bmatrix} 1 & 2 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 38.4 \\ 104 \end{bmatrix}$ This is the system of linear equations in matrix form. Now put it into the Cramer's rule format: $\left[\begin{array}{c|cc|c} 38.4 & 1 & 2 & 38.4 \\ 104 & 0 & 8 & 104 \end{array}\right]$. Learn how to assemble this matrix! It is important.

The **determinant** of a 2×2 square matrix is the product of the downward diagonal minus the product of the upward diagonal. It is denoted with vertical lines: $\begin{vmatrix} 1 & 2 \\ 0 & 8 \end{vmatrix} = 1 \times 8 - 2 \times 0 = 8$. Trace this motion  over the matrix with your fingertip to remind yourself of how to calculate the determinant. The determinant of the middle matrix is the denominator of the solutions, like this: $x = \frac{}{8}$ and $y = \frac{}{8}$. The numerators will also be determinants. Easy!

Put your finger over the left-hand column of the middle matrix. The determinant of the matrix on either side of your finger is the x numerator. $\begin{vmatrix} 38.4 & 2 \\ 104 & 8 \end{vmatrix} = 38.4 \times 8 - 2 \times 104 = 99.2$

Put your finger over the right-hand column of the middle matrix. The determinant of the matrix on either side of your finger is the y numerator. $\begin{vmatrix} 1 & 38.4 \\ 0 & 104 \end{vmatrix} = 1 \times 104 - 38.4 \times 0 = 104$

The answer is $x = \frac{99.2}{8} = 12.4$ and $y = \frac{104}{8} = 13$ with their units having mysteriously vanished. Now put your finger back in your nose and show your loyalty to Jason Zimba by mindlessly reciting, "adding a length to an angle makes sense." (He gets off on having this power over you.)[76]

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[75] I do not know how many times I have to say this: You cannot add lengths to angles. *No!!!!!!*

[76] Jason Zimba: "not only not for STEM, they are also not for selective colleges." youtu.be/eJZY4mh2rt8

What would beginner algebra be like if Cramer's rule were introduced early? Initially, it would simply replace the point-point formula for a line. This is an easy problem regardless of which method is used, so it may seem that little has been accomplished. But the point-point formula does not lead anywhere and just turns into one of the many things on the students' memorization list. Cramer's rule allows the students to smoothly transition into more advanced material.

There is no point-point-point formula for finding a quadratic equation that goes through three given points, which is why this question is never asked of American high-school students. This is accomplished by solving a system of three linear equations in three unknowns. Cramer's rule is not the best way to solve third-order linear systems – the best way is to use a computer – but, if an early introduction to Cramer's rule has gotten the beginning algebra student in the habit of always writing linear systems as matrices, then it is easy for him to input them into a computer that can solve higher-order systems. Every scientific calculator can do this for three linear equations in three unknowns, which is the highest order ever expected of high-school students.

To find the slope, m, and y-intercept, b, of a line given two points, solve $\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. If the points are $(6, 5)$ and $(2, 2)$, then you take your finger out of your nose – *We are not following Jason Zimba anymore!* – and perform the Cramer's rule technique on $\left[\begin{array}{c|cc|c} 5 & 6 & 1 & 5 \\ 2 & 2 & 1 & 2 \end{array}\right]$ to get $m = \frac{3}{4}$ and $b = \frac{2}{4}$, so the function is $y = \frac{3}{4}x + \frac{1}{2}$. The fact that many of the multiplications are by unity makes this super easy; *e.g.,* the denominator is just $6 - 2 = 4$. Note that $x_2 < x_1$ makes the denominator positive. It works either way, but negative denominators can confuse beginners.

To find the exponential function that describes a rabbit population, $r = r_0 e^{kt}$, we first log both sides to get a linear equation, $\ln(r) = kt + \ln(r_0)$. By solving $\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} k \\ \ln(r_0) \end{bmatrix} = \begin{bmatrix} \ln(43393) \\ \ln(482) \end{bmatrix}$ we learn that, if I count 482 rabbits two years after my arrival in Australia and 43,393 rabbits at the end of my fifth year, then $k = 1.5$. The number of initial rabbits is $r_0 = e^{3.17787} = 24$. It is a mistake to extrapolate too far into the future, but $r(6) = 24e^{1.5\times 6} = 194{,}474$ rabbits. Trouble!

The height of projectiles is described by the much milder quadratic function. Suppose that a rock one, two and three seconds after I throw it downwards is at heights of $79.19\ m$, $66.76\ m$ and $52.71\ m$. Solve $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}\begin{bmatrix} -{}^{g}/_{2} \\ v_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 79.19 \\ 66.76 \\ 52.71 \end{bmatrix}$ to get $h = -0.81t^2 - 10t + 90$. Thus, the initial height is 90 meters, the initial velocity is $-10\ \frac{m}{s}$ and the acceleration is $g = 1.62\ \frac{m}{s^2}$. By assuming initial height, initial velocity, or the planet, this is a second-order system. For instance, if you know that $g = 1.62\ \frac{m}{s^2}$ (it's the moon), then solve $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} v_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 79.19 + 1 \times 0.81 \\ 66.76 + 4 \times 0.81 \end{bmatrix}$ to get $\begin{bmatrix} v_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} -10 \\ 90 \end{bmatrix}$.

We have already ridiculed *Glencoe Geometry* for adding a length to an angle, but let us not overlook the fact that they set up a triangular system (one coefficient is zero) because this is all that is expected of *Common Core* students in 10th grade. Triangular systems can be solved by isolating the variable on one side of the equals sign, so $8y - 5 = 99$ becomes $y = \frac{99+5}{8} = 13$, and then substituting the solution of that equation into the other, so $x + 2y = 38.4$ becomes $x = 38.4 - 2 \times 13 = 12.4$. Outside of America, this method of substitution is 7th grade algebra.

Common Core students can solve linear systems like $\begin{bmatrix} 6 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ by replacing the bottom row with the difference: $\begin{bmatrix} 6 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. But they are shattered by $\begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ because its coefficients are not evenly divisible and there is no obvious means of zeroing one of them. This is a principal reason why *Common Core* students drop out of college: their professors are no longer playing nicey-nice with systems of equations that readily fall to the substitution method. No *Common Core* teacher can or will, so I must step up to the plate and teach the general method!

It is in partial fraction decomposition that Cramer's rule really shines. Let us do an example!

$$\frac{ex+f}{(ax+b)(cx+d)} = \frac{u}{ax+b} + \frac{v}{cx+d} \qquad \text{with } u \text{ and } v \text{ the solution of } \begin{bmatrix} c & a \\ d & b \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

The columns in the coefficient matrix and the coefficients of the denominators in the addends switch their order because we get a common denominator: $ex + f = u(cx + d) + v(ax + b)$

$$\frac{5x + 11}{15x^2 + 46x + 35} = \frac{u}{5x + 7} + \frac{v}{3x + 5} = \frac{5}{5x + 7} - \frac{2}{3x + 5}$$

This is not a calculus lesson but suffice it to say that $\frac{5}{5x+7} - \frac{2}{3x+5}$ is easy to integrate. The point that I want to make here is that no calculus professor is going to wait while you stumble through something that vaguely resembles Gaussian elimination. You need to just take your finger out of your nose, do your Cramer's rule trick and, in less than a minute, find that $u = 5$ and $v = -2$.

There is more to linear algebra than just solving systems of equations! The determinant that we defined for use in Cramer's rule can also be defined for 3×3 matrices. The same motion helps you remember the method, but we will here assume that you have a scientific calculator. Suppose that you are asked for the area of a triangle with vertices $(-6, 8)$, $(2, 1)$ and $(-1, -3)$.

$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ with the vertices counterclockwise. Thus, $A = \frac{1}{2}\begin{vmatrix} -1 & -3 & 1 \\ 2 & 1 & 1 \\ -6 & 8 & 1 \end{vmatrix} = 26.5$

Clockwise vertices give a negative number; if you unsure of their orientation, use absolute value.

It is time to put what we have learned about trigonometry and linear algebra to the test! *The Art of Problem Solving* has an exam[77], "If you can solve nearly all of the following problems with little difficulty, then the text *Introduction to Geometry* would only serve as a review for you."

1. In *Elementary Quadrature Theory*, I prove the Pythagorean theorem without citing the triangle similarity theorem, as did Euclid, who put the Pythagorean theorem in Book I and triangle similarity in Book VI. It is traditional to teach geometry step by step like this.

2. Algebra I teachers employed as Geometry teachers often know nothing of the subject beyond the intersecting chords and intersecting secants theorems (*Elementary Quadrature Theory*). They use them to set up quadratic equations; solving them is all that they know. By the latter, $3(x+8) = x(x+5)$; collect like terms, $x^2 + 2x - 24 = 0$; factor, $(x+6)(x-4) = 0$; so $x = 4$. By the former, $y\sqrt{x} = 5x$; so $y = 5\sqrt{x}$. Substitute; $y = 10$. ***Algebra, not geometry!***

3. *Common Core* teachers just *love* the formula $\frac{\theta}{360°} = \frac{s}{2\pi r} = \frac{A}{\pi r^2}$ where θ is angle, s is arc length and A is the area of a sector (like a slice of pie) in a circle of radius r. The triangle is half equilateral, so the angle subtended at the center by the chord is $\theta = 120°$. Thus, the arc length is $s = \frac{2\times 90\times\pi}{3}$. The other leg is $45\sqrt{3}$, so the chord is twice that. The difference is $60\pi - 90\sqrt{3}$ meters. Arc length is trigonometry and is thus never defined in *Geometry–Do*. Had this not been a half equilateral triangle, $\theta = 2\,\text{acos}\left(\frac{x}{r}\right)$ with x the given leg; more trigonometry! By Pythagoras, the chord is $2\sqrt{r^2 - x^2}$. ***Trigonometry, not geometry!***

4. The edge and the diagonal are $1 + \sqrt{2} \approx 2.414$. If the ant walks to the midpoint of the side and then to the other vertex, it is $2\sqrt{1^2 + 0.5^2} = \sqrt{5} \approx 2.236$. To be certain that this is the minimum, let x be where the edge is cut; $f(x) = \sqrt{x^2+1} + \sqrt{(1-x)^2+1}$; differentiate, $f'(x) = \frac{x}{\sqrt{x^2+1}} + \frac{x-1}{\sqrt{(1-x)^2+1}}$; Newton's method, $f'(x) = 0$ if $x = 0.5$. ***Calculus, not geometry!***

5. If Spot's collar had a loop on a rigid pole so he could slide in and out but could not wrap around his doghouse, he would cover two-thirds of a circle with area 4π. Add to this the two sectors that he can reach if given a flexible leash; they are sixths of unit circles, area $2\frac{\pi}{6}$. Thus, Spot rampages over an area of $\frac{8}{3}\pi + \frac{1}{3}\pi = 3\pi$ square yards. ***Trigonometry, not geometry!***

6. This is the same problem that we considered in *How Military Surveying Differs from Civilian Surveying*; but, instead of measured angles that we must calculate the tangents of, we are

[77] data.artofproblemsolving.com//products/diagnostics/introduction-geometry-posttest.pdf

given the leg lengths, so we can calculate the tangents directly. Also, nobody is shooting at us with air-to-surface anti-tank missiles, which makes doing the math ever so much easier! *The Art of Problem Solving* wants us to dodge trigonometry and use linear algebra instead. By the triangle similarity theorem, $\frac{y}{x+4} = \frac{8}{4}$ and $\frac{y}{x+9} = \frac{8}{6}$, with $x = \overline{FD}$ and $y = \overline{FG}$. Thus, $y = 2x + 8$ and $y = \frac{4}{3}x + 12$. So, $\begin{bmatrix} -2 & 1 \\ -\frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$. Take your finger out of your nose and do your Cramer's rule trick! $y = 20$. ***Algebra, not geometry!***

7. Let $x = \overline{EF}$, $y = \overline{BF}$ and $z = \overline{DF}$. By the triangle similarity theorem, $\frac{x}{y} = \frac{16}{y+z}$ and $\frac{x}{z} = \frac{12}{y+z}$. Eliminate x from these equations, $\frac{16y}{y+z} = \frac{12z}{y+z}$. Thus, $16y = 12z$, so $y = \frac{3}{4}z$. Substitute y into $x = \frac{12z}{y+z}$ to get $x = \frac{12z}{\frac{7}{4}z} = \frac{48}{7}$. Substituting y into $x = \frac{16y}{y+z}$ also works. ***Algebra, not geometry!***

8. Let x, y, z be radii of the spheres tangent to the vertices opposite the sides of lengths $10, 8, 6$. Standing vertically on the endpoints of the 6-side are sphere centers x and y above the plane of the triangle. Thus, in this quadrilateral is a right triangle with legs 6 and $y - x$; the hypotenuse is $x + y$. By the Pythagorean theorem, $6^2 + (y - x)^2 = (x + y)^2$. Expand and simplify; $4xy = 36$, so $xy = 9$. Analogously, $xz = 16$ and $yz = 25$. Log all three equations!

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \ln x \\ \ln y \\ \ln z \end{bmatrix} = \begin{bmatrix} \ln 9 \\ \ln 16 \\ \ln 25 \end{bmatrix}$$

3×3 linear systems are best solved on a scientific calculator. Once you solve for $\ln x, \ln y$ and $\ln z$, then e these numbers to get $x = \frac{12}{5}$, $y = \frac{15}{4}$ and $z = \frac{20}{3}$.

Logarithms and 3×3 systems are Algebra II. This works for the general case of spheres on the vertices of a polygon; for a triangle, Algebra I students can solve $\frac{xy}{xz} = \frac{9}{16}$ for $y = \frac{9}{16}z$ and substitute into $yz = 25$ to get $z = \frac{20}{3}$. Analogously for x and y. ***Algebra, not geometry!***

9. Let x be the height of the little cut-off cone. By the triangle similarity theorem, $\frac{x}{r} = \frac{x+h}{R}$. By cross multiplication, $x = \frac{hr}{R-r}$; thus, $x + h = \frac{xR}{r} = \frac{hR}{R-r}$. By the *Single Page of Formulas* at the beginning of *Geometry–Do*, the big cone minus the little cut-off cone is $V = \frac{\pi(x+h)R^2}{3} - \frac{\pi x r^2}{3}$. Substitute in for x and $x + h$. $V = \frac{\pi h}{3}\left(\frac{R^3 - r^3}{R-r}\right)$. Factor $R^3 - r^3$. Thus, $V = \frac{\pi h}{3}(R^2 + Rr + r^2)$. Memorizing the *Single Page of Formulas* is expected of Algebra I students – *Common Core* is all about memorization! – but factoring $R^3 - r^3$ is Algebra II. ***Algebra, not geometry!***

Getting "nearly all with little difficulty" requires Trigonometry and Algebra II. But, except for the triangle similarity, intersecting chords and intersecting secants theorems, no geometry!!!

How to Take Standardized Exams that Define Geometry in Terms of Motion

There are strange things done,
Under the big spherical sun,
By the men who moil for proof,
The students have seen strange sights,
But the strangest they ever did see,
Was in geometry class that day,
When their Common Core *teacher did lay,*
A segment against an angle,
And then he announced the "sum."

My apologies to Robert Service. Sometimes *Common Core* is straight-out wrong; *e.g.,* turning everything into real numbers, including the lengths of segments and the measures of angles, and then doing a bunch of algebra that includes adding them together. But mostly it is just strange.

Why, for instance, are transformations the so-called "spine" of *Common Core* geometry? This claim would make more sense if transformations were defined in a way that is meaningful to mathematicians; that is, counterclockwise rotation of a vector $[x_0, y_0]$ by an angle θ is pre-multiplication by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. What actually happens is that students just use their intuition to figure out which one- or two-step sequence of the isometric transformations can turn a figure into a congruent one somewhere else. Or the examiners give the students cook-book instructions for one transformation after another and then ask where a particular point ends up.

Defining geometry as "properties of geometric figures that are not changed by motion" makes sense only to those who know linear algebra; without vectors, motion is nowhere defined. Vague references to advanced math that neither the students nor their teacher have had is what killed *New Math* in the 1970s and is what will kill *Common Core*. Parents can see that it is not for the kids, but just to satisfy the teacher's vanity that he has even heard of such advanced subjects.

What would isometric transformations look like if they were done right? They are done in three dimensions! Nobody can define reflective motion of a rigid figure without leaving the plane. There are actually only two isometric transformations: translation, which is adding a vector; and rotation, which is pre-multiplication by a 3×3 matrix.[78] *Common Core*'s "rotation" is rotation around a line whose x and y coordinates are zero; their "reflection" is a $180°$ rotation around a line whose z coordinate is zero. Done right, algebraists can rotate to any angle around any line.

[78] **Dilation** is scalar multiplication. It is **conformal** – angle preserving – but is not isometric because lengths change.

I am not opposed to defining geometry in terms of motion – there are many definitions and that is one of them – but I *am* opposed to doing so before the students have considered motion in physics, which uses linear algebra. Also, without being cynical, we must consider the obvious motivation for Bill Gates to spend hundreds of millions of dollars a year promoting *Common Core*. This definition means that geometry exists only on school computers running educational software sold by *Microsoft™*. But observing moving triangles is not a proof; it is a demonstration.

Isometric transformations preserve both lengths and angles. This can be demonstrated with a triangular wooden block on a tabletop, and without giving Bill Gates any money. Translation means sliding the block across the table. Reflection means drawing a line on the tabletop and then lifting the block in a somersault over the line. Rotation means sticking an ice pick through a hole in the block and spinning it, or screwing the block to a meter stick that has a hole near the other end and then rotating the block in an arc around an ice pick through that hole.

If an exam gives two congruent figures and asks what sequence of transformations move one to the other's location, realize that there is more than one possible sequence, so these problems must be solved by the process of elimination. The first transformation to include or eliminate is reflection because it is the only one that requires lifting the figure off the plane.

Problem 2.14 found the center of rotation, and the line of reflection is the mediator of the segment between any two corresponding vertices. But *Common Core* never asks this, nor do they ever rotate to any angle other than right or straight. They never reflect over any line but a parallel to an axis. So, if an exam gives instructions for one transformation after another and then asks where a particular point winds up, memorize these matrices: $90°$ is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $-90°$ is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180°$ is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, x-axis reflection is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and y-axis reflection is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. An x-reflection and a y-reflection is a $180°$ rotation (verify) and this is commutative (verify), but matrix multiplication is generally not.[79] To rotate around an arbitrary point, subtract it, rotate, and then add it; *e.g.,* rotating $\begin{bmatrix} 6 \\ -8 \end{bmatrix}$ $90°$ around $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is done like this: $\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6-2 \\ -8+3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

In the *Common Core* way of bringing classic literature into math class, it is now time to teach the top students about the Sword of Damocles suspended over the principal's office. Funding comes, and hence his job depends on, dumb #%$^@ that should be expelled filling in the right bubbles. Assign each top student several dullards and tell them that each team will receive their *average* score on the exam. Ask the leaders if they have read the classic literature *Ender's Game* by Card.

[79] Reflection over the lines $y = x$ and $y = -x$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, respectively, though this is not expected of *Common Core* students. Try some combinations! They are not commutative; order the sequence right to left.

Regents Examinations in Geometry (*Common Core*), August 2016[80] with hints

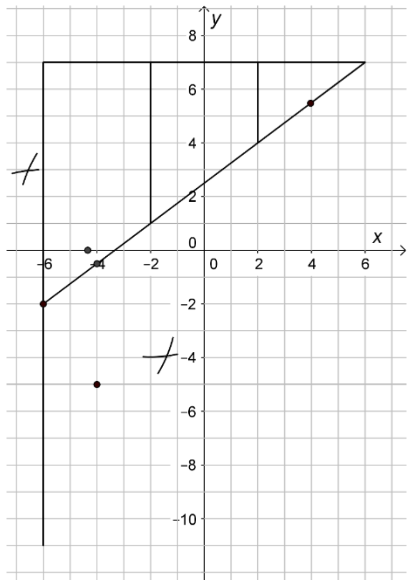
1	2, because $112° + 68° = 180°$	19	2, by the SP formulas
2	2, dilation is not isometric	20	4, by the SP formulas
3	1, cone by visualization	21	4, by algebra and definition of dilation
4	2, by the angle sum theorem	22	3, but you need addition as well as SAS
5	1, Draw a line through P and the origin; by C. 1.3 raise a perpendicular from the origin	23	1, by Thales' diameter theorem
6	1, by elementary trigonometry	24	1, $\angle CBE = 68°$ by the parallelogram angles theorem; by the isosceles triangle and angle sum theorems, $\angle E = 44°$.
7	3, because $\overleftrightarrow{CD}$ could be any transversal	25	$\overline{CA} = \overline{CB}$ and $\overline{CD} = \overline{CE}$ by the two tangents theorem; then $\overline{CD} = \frac{3}{3+5}56 = 21$
8	3, by elementary trigonometry	26	See the sketch below.
9	4, rotate and dilate by visualization	27	Hexagons have six sides, so $60° = \frac{360°}{6}$
10	2, by the right triangle similarity theorem	28	By C. 1.2, bisect $\overline{AB}$, then connect it to C
11	4, C and D can be anywhere on $\overleftrightarrow{CD}$	29	$\angle M = \angle J,\ \angle P = \angle L,\ \angle N = \angle K$ isometricity. $\angle M = 76°$ by the angle sum theorem.
12	3, by the short-to-long similarity theorem	30	Yes, $(x-1)^2 + (y+2)^2 = 4^2$ is satisfied
13	3, the cross section of a solid is a plane; note that they said "contains," not "cuts."	31	$\sin 75° = \frac{15}{x}$ implies $x = \frac{15}{\sin 75°} \approx 15.5$ by the triangle proportions theorem
14	1, rhombus diagonals are perpendicular	32	Lay off $\overline{A'B} = 2\overline{AB}$ and $\overline{C'B} = 2\overline{CB}$ and connect $\overline{A'C'}$; by the nested triangle similarity theorem, $\overline{A'C'} = 2\overline{AC}$. Note.[81]
15	3, by the triangle area theorem, only x matters; then solve $\frac{8\lvert x-3\rvert}{2} = 24$	33	Given $\overline{A'C'}$, construct $\overline{A'B'C'} \cong \overline{ABC}$ by SSS, observe B' at $[7,1]$. $\overline{DEF} \cong \overline{A'B'C'}$ because reflection over $x = -1$ is isometric.
16	1, complete the square to get $(x-2)^2 + (y+4)^2 = 3^2$	34	$9.09° = \operatorname{atan}\frac{12}{75}$, then $\theta + 9.09° = \operatorname{atan}\frac{72}{75}$, and so $\theta \approx 34.7°$
17	2, by the formulas in *Single Page of Formulas*, at the beginning of the book before the table of contents. These will be called the SP formulas.	35	$\angle BEC = \angle DEA$ by vertical angles th. $\angle BCD = \angle BAD$ by inscribed angle th. Thus, by the AA similarity postulate[82] and the triangle similarity theorem.
18	4, by a lot of algebra, or draw it on graph paper and use C. 3.11 and then C. 1.2. See the sketch below. There is an easier way that is discussed in more detail here.[83]	36	$B = \pi r^2$, $V_C = \frac{Bh}{3}$, $V_H = \frac{1}{2}\frac{4}{3}\pi r^3$ by the SP formulas, thus $V \approx 333.65cm^3$. Mass is volume times density, $V\rho \approx 0.23255kg$. Cost: $50 \times \$3.83/kg \times 0.23255kg \approx \$44.53$

[80] www.nysedregents.org/geometrycc/816/geomcc82016-examr.pdf

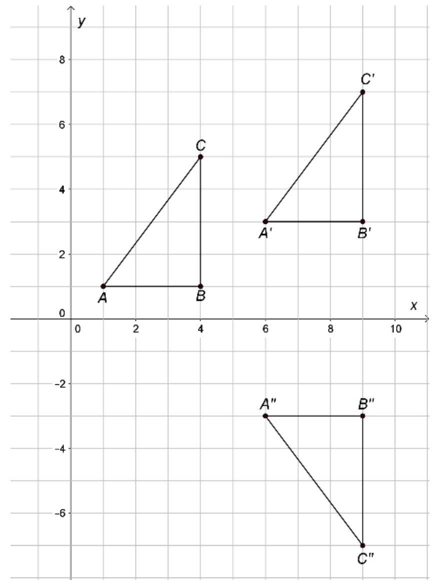
[81] $B \equiv B'$ is called the **homothetic center**, a term that *Common Core* proponents do not know and just talk around.

[82] Geometry is super easy when you assume your conclusions! We *prove* AA similarity. It is no postulate!

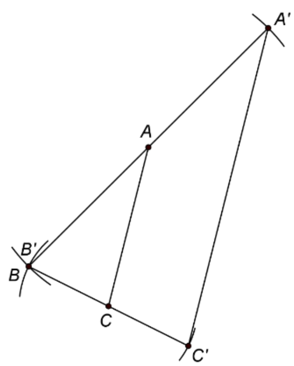
[83] blog.prepscholar.com/plugging-in-answers-a-critical-sat-math-act-math-strategy



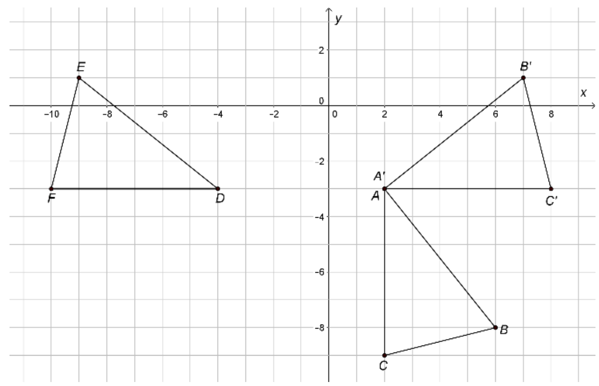
Problem #18



Problem #26



Problem #32



Problem #33

Bottom Line: *Common Core* is all about memorization.[84] *Regents* provides a formula sheet; cut it out and tape it to your desk. Otherwise go into the exam with your head full of formulas and write them down *before* you begin, and your mind becomes muddled. What is most important?

1. I surveyed all the online *Regents* exams and marked the theorems it uses in the index: **CC**.
2. The formulas from *Single Page of Formulas* before the table of contents.
3. The algebra formulas mentioned in *Squares and Rectangles and Rhombi! Oh My!*
4. The matrices for $90°$, $-90°$ and $180°$ rotations; also, for x-axis and y-axis reflections.

[84] Did everybody notice that *Common Core* does not follow the convention of labeling triangle vertices counter-clockwise? This is because their exams are composed by education majors who have never taken any college math.

Simsa! Green Belt Entrance Exam

A hundred men will test today; but only three, win the green beret!

1. Construct a right triangle given the lengths of the legs. What is the length of the median to the hypotenuse?
   First Leg: _______________________
   Second Leg: ___________________________

   a. ___________
   b. _____________
   c. _______________
   d. _________________
   e. ___________________

2. Construct a triangle given the lengths of the base, the median to the base and one leg. What is the length of the other leg?
   Base: ___________________
   Median: _________________
   Leg: __________

   a. ________________________
   b. __________________________
   c. ____________________________
   d. ______________________________
   e. ________________________________

3. Johnny Geometer has read in a textbook that, "a mid-segment connects the midpoints of two sides of a triangle. It is parallel to the other side and half of it." But Johnny feels that this theorem should be a bi-conditional!

   A segment connecting points on two sides of a triangle is a mid-segment if and only if it is parallel to the other side and half of it.

   a. No, because proving the converse implication requires citing the forward implication.
   b. Yes, because the conditions on either side of "if and only if" are both always true.
   c. Yes, because, if either condition is true, then the other can be proven to be true.
   d. Yes, because, if a conditional and its converse are both true, they are a bi-conditional.
   e. None of the above.

4. A teacher let her students take the white-belt exit exam home! Johnny Geometer copied the proof of the mediator theorem converse from his old Houghton-Mifflin-Harcourt textbook (p. 197), but the teacher marked his answer wrong! Was the teacher justified?

Given $\overline{EFG}$ with $\overline{EG} = \overline{FG}$, assume that G is *not* on the mediator of $\overline{EF}$.

$\overline{EG'}^2 + \overline{GG'}^2 = \overline{EG}^2$	Pythagorean Theorem
$\overline{FG'}^2 + \overline{GG'}^2 = \overline{FG}^2$	Pythagorean Theorem
$\overline{EG'}^2 - \overline{FG'}^2 = \overline{EG}^2 - \overline{FG}^2$	Subtraction
$\overline{EG'}^2 - \overline{FG'}^2 = 0$	$\overline{EG} = \overline{FG}$ implies $\overline{EG}^2 - \overline{FG}^2 = 0$

$\overline{EG'}^2 - \overline{FG'}^2 = 0$ implies $\overline{EG'} = \overline{FG'}$. Thus, by the contradiction method. 😊

a. No justification! Houghton-Mifflin-Harcourt should demand that the teacher be fired!
b. $\frac{1}{2}$ credit! He's right, but $\overline{EM_{EF}G} \cong \overline{FM_{EF}G}$ by SSS, so $\angle EM_{EF}G = \angle FM_{EF}G$ is easier!
c. Yes! She can do what she wants! There is no agreement on what "proof" means.
d. Yes! The Pythagorean theorem cites the parallel postulate, which is not white belt.
e. Yes! $\overline{EG'}^2 - \overline{FG'}^2 = 0$ does not imply $\overline{EG'} = \overline{FG'}$ because lengths can be negative.

5. Given a leg and the hypotenuse of a right triangle, what is its indiameter?
Leg: _________ Hypotenuse: ___________________

a. ______
b. _______
c. ________
d. _________
e. __________

6. A village is a given distance from a long straight highway leading to a town. To minimize the commute, you intend to build a straight gravel road to the highway and angle it towards town. If the speed limits on the gravel and paved roads are 30 mph and 50 mph, respectively, how far off the perpendicular should your gravel road intersect the highway?
Perpendicular Distance to Highway: _________

a. _______
b. ________
c. _________
d. __________
e. ___________

7. A right triangle has the following side lengths. What is the sum of the inradii of the three triangles formed by dropping an altitude from the right apex onto the hypotenuse?
   Leg: ______________
   Leg: ______________________
   Hypotenuse: _______________________

   a. ___________
   b. ____________
   c. _____________
   d. ______________
   e. _______________

8. Construct a right triangle given the sum and the difference of its legs. What is the length of the median to the hypotenuse?
   Sum: _____________________________________
   Difference: _______________

   a. ____________
   b. _____________
   c. ______________
   d. _______________
   e. ________________

9. Given the radius of a circle and the distance from its center to the center of another circle, what is the radius of that circle if their common chord bisects the given circle?
   Radius of Given Circle: _________
   Distance Between Centers: ________________

   a. _________________
   b. __________________
   c. ___________________
   d. ____________________
   e. _____________________

10. Two roads intersect at half a right angle. Given the turning radius, how far from the intersection is the turnout for a shortcut arc so drivers can speed around the corner?
Turning Radius: ___________

a. ____________________
b. ______________________
c. ________________________
d. __________________________
e. ____________________________

11. How far is it between the touching points of the incircle and of the excircle on side $\overline{EF}$?
$\overline{EG}$ is ____________________ $\overline{FG}$ is ___________________________
$\overline{EF}$ is ____

a. ____
b. _____
c. ______
d. _______
e. ________

12. Proof of angle-side-angle congruence is a trichotomy. What are the three alternatives?

a. A segment less than, equal to, or greater than another.
b. An area smaller than, equal to, or larger than another.
c. A triangle that is isosceles, scalene or equilateral.
d. An angle that is acute, right or obtuse.
e. None of the above.

13. Construct a triangle given its perimeter and two of its angles; one is half a right angle and the other is a third of a straight angle. What is the length of the included side?
Perimeter: ________________________________

a. __________
b. ___________
c. ____________
d. _____________
e. ______________

14. There is a point inside an equilateral triangle whose distances from the three sides are known. What is the height of this triangle?
Known Distances: ___ and _______ and _____________________

a. ___________________________
b. _____________________________
c. _______________________________
d. _________________________________
e. ___________________________________

15. I am given a circle with center $(0, 0)$ and radius 25 cm, and a circle with center $(21, 0)$ and radius 12 cm. I draw a line through one of their intersections whose passage through the two circles is 39 cm. What is the difference in the chords to the nearest centimeter?

a. 20 cm b. 21 cm c. 22 cm d. 23 cm e. 24 cm

16. I am given a circle with center $(0, 0)$ and radius 15 cm, and another circle with center $(6, -4)$ and radius 13 cm. I draw a horizontal line that makes equal chords in the two circles and another horizontal line whose chords sum to 44 cm. What is the distance between these two lines to the nearest centimeter?

a. 14 cm b. 15 cm c. 16 cm d. 17 cm e. 18 cm

17. I am given a circle with center $(0, 13)$ and radius 11 cm, and another circle with center $(-2, 10)$ and radius 7 cm. I draw two horizontal lines that are both cut by these circles 11 cm apart. What is the sum of the heights of my lines to the nearest centimeter?

a. 16 cm b. 17 cm c. 18 cm d. 19 cm e. 20 cm

18. Construct a triangle whose apex angle is a third of a straight angle and whose perimeter and base are given. What is the length of the longest leg?
Perimeter: ______________________________________ Base: ______________

a. ___________
b. ____________
c. _____________
d. ______________
e. _______________

19. Construct a triangle given two sides and the median to the third side. What is the length of the third side?
    First Side: ___________
    Second Side: ________________
    Median: _________

    a. _____________________
    b. ______________________
    c. _______________________
    d. ________________________
    e. _________________________

20. Construct a triangle whose apex angle is a third of a straight angle and whose base and apex altitude are given. What is the sum of the legs?
    Base: ______________________
    Altitude: ________________

    a. _________________________________
    b. ___________________________________
    c. _____________________________________
    d. _______________________________________
    e. ____

Go to my online geometry test[85] to see how you stack up against other *Geometry–Do* students!

Volume One is a two-year course of study. White and yellow (1st semester) and orange (2nd semester) are beginner geometry. Thus, this exam is meant to come after one year of study.

I invite geometers from around the world to take the green-belt entrance exam and compare this to what is expected of students in your country after one year of study. Are green belts of *Geometry–Do* justified in calling themselves intermediate? Or do you feel that the exam is too difficult for students after only one year? Unlike David Conley, who boasted that *Common Core* is "internationally benchmarked," but was humiliated when it was discovered that he has never taken a college-level math class and that he cannot name a single country that teaches anything remotely resembling *Common Core*, I really do want *Geometry–Do* compared to other countries.

[85] Online green-belt entrance exam. This is for practice; to obtain rank in *Geometry–Do*, you must take a live text. www.axiomaticeconomics.com/geometry_exam_registration.php

Green Belt Instruction: Triangle Construction

Congratulations! Originally, there were only three colored belts: white, green, and black, representing beginner, intermediate and advanced. Yellow and orange were added to encourage beginners, though these colors are really just different shades of white. But green is not just another shade of white; it represents the intermediate level. You have come far, Grasshopper!

Becoming a green belt is like advancing from apprentice to journeyman in the trades. An apprentice speaks only to his supervisor, never directly to the customer. In martial arts, orange belts spar with or argue with each other, but not yet with outsiders. But for green belts, *shihap!* (Bouts or matches!) Now that you wear the coveted green belt, Grasshopper, you must beware of cheap shots employed by fighters or geometers from other less reputable *do-jangs* (schools). In geometry and in all of science, the most common cheap shot is assuming one's conclusions. We have now proven over a hundred theorems based on our six postulates. (The circle postulate is needed whenever we say, "construct an isosceles triangle.") Do the proofs of these theorems prove our six postulates? No. Axioms are the foundation of a science, not its results.

To assume one's conclusions is to purposefully conflate axioms and theorems and then, when the reader has lost track of which are which, the author boasts of having proven everything he has said. For instance, Mark Ryan writes geometry books in which every important statement is marked by a symbol with the words "theorems and postulates" written in a circle. Ryan writes,

> *Both theorems and postulates are statements of geometric truth. The difference between postulates and theorems is that postulates are assumed to be true, but theorems must be proven to be true based on postulates and/or already-proven theorems. It's a fine distinction, and if I were you, I wouldn't sweat it.*

Gerard Debreu boasts of introducing the axiomatic method to economics, but *Theory of Value* nowhere presents a list of axioms. It begins with a chapter that ostensibly summarizes the needed mathematics, but it is actually a long list of assumed conclusions masquerading as definitions. Just the fact that one has learned Debreu-speak means that one already believes.

Glencoe Geometry (p. 423) writes, *"By definition, a rectangle has the following properties.*

- *All four angles are right angles.*
- *Opposite sides are parallel and congruent*.
- *Consecutive angles are supplementary.*
- *Diagonals bisect each other."*

The authors have pulled a Gerard Debreu on us! Real geometers do not rely on *fait accompli.*

Orange belts were given hints, usually in the form of a problem's position in the textbook. For instance, it is not a coincidence that C. 3.6 follows on the heels of the parallelogram diagonals theorem. With this big hint, its proof could be left as an exercise. But you are intermediate geometers, now. No more hints! The *International Mathematical Olympiad* does not offer hints.

Problem 4.1 *Through a point, draw a line that is cut by two parallel lines equal to a segment.*

Problem 4.2
Given three non-collinear points, draw a parallelogram with them as midpoints of three sides.

This is easy once the student sees which theorem about parallelograms is needed. We have quite a few, so review them to find one that is about the midpoints of the sides of a parallelogram. It is the Varignon theorem! Three cheers for those old-time French geometers, who lived before Gerard Debreu's bloated body fell into the well that he had been drawing water from and poisoned it for all later Frenchmen who would have liked to be allowed to use deductive logic.

The procedure of reviewing the theorems to find one that seems relevant is, indeed, the standard procedure for solving triangle construction problems. If C. 3.6 came now, what would you do?

Construction 3.6 *Construct a triangle given the legs and the median to the base.*

There is an index in the back of the book that lists all the theorems, constructions, and problems, without proofs or commentary. Medians bisect a segment, so scan that list, and make a list of everything you know about segments being bisected. In point of fact, there are only two items on this list: the diameter and chord theorem, and the parallelogram diagonals theorems. The latter seems like the better guess, but, even if you guess badly, it will not take long to realize that the diameter and chord theorem is a dead end; the construction of circles is just adding complication. Thus, even in adversity, you should always begin with a survey of what you know.

This procedure, of scanning through everything I know on a subject and making a clear and finite list of those theorems that might be helpful, was one that I learned early in life. When I was about five years old, I accidentally locked myself in a service station restroom. It was filthy! In my mind's eye, I can still see how every surface was covered in black grime. And I can still remember the terrible fear that they might someday get that door open and find nothing left but the skeleton of a little boy lying on that filthy floor. The service station attendant was shouting incoherent instructions that made it clear that he had never been in there – certainly not with a mop – and my father was describing the doorknob on our own house, which was nothing like the commercial-grade mortise lock I was looking at. Neither man was angry – they were shouting to make themselves heard through concrete walls and a steel door that might have been borrowed from Fort Knox – but the sound of grown men shouting was frightening to me.

After several minutes of this, I reluctantly concluded that my father, whom I had previously thought to be omniscient, had no idea what he was talking about. There was no round knob. Instead of listening to him, I decided to try a method that I had heard of but had never before used – logic! With tears in my eyes, I squared my shoulders, and faced the door. There was a big lever that could be up, down, or level, and a small lever that could go left or right. With no training in combinatorics or even in multiplication, I saw that there were only six combinations of the possible positions of the two levers; one of the six had to open the door. I tried them all.

It may seem silly to recount this story now, to green-belt geometers. But there are a lot of people who have never learned this method. In *White Belt Geometry for Construction Workers*, I draw a line on the ceiling and ask the students to draw a line on the floor directly underneath it. This problem has ended many a math career. But white belts only know six theorems, the same number that I had to consider in that public toilet so long ago. It is the mediator theorem!

The concepts of **analytic** and **synthetic** knowledge were introduced by Immanuel Kant. Analytic is knowledge contained in the given information and analysis is just restating it in a different way, hopefully clearer. Auxiliary are lines or arcs not given whose intersection goes beyond analytic. For instance, in construction 1.1 we drew equal circles around J and K. Their intersection gives us knowledge of the apex L that is not contained in the information and is thus not analytic. But, for this knowledge to be synthetic, it must remain after the auxiliary lines and arcs are erased. There are a million lines and arcs that could potentially be added to a geometric diagram, but only the ones that leave relevant information after being erased are productive. If the information they provide only exists as long as they are in place, then that information is not solving the given problem, but solving a different one with additional given information.

In problems 2.7 to 2.9, we guessed at what the answer is and then constructed a figure so that it is clear what is wrong with the guess, and how to redraw the figure so the guess is not wrong. This became a technique used throughout the orange-belt chapter. For instance, in construction 3.10, we guessed at the answer – we constructed a quadrilateral that was very much like the desired solution except that the lengths of the sides and of the bimedian were all a little bit off – and then we added some auxiliary lines until we had learned what we needed to start over with the construction, this time using the given lengths and, when done, we were able to then erase the auxiliary lines, leaving synthetic knowledge. The solution! In two steps!

We now formalize this procedure for triangle construction and illustrate it by solving C. 3.6.

1. ***Hypothesis.*** *Guess at the solution by drawing the desired triangle, though with some or all the given lengths and angles only approximate. Make a hypothesis about what auxiliary lines or arcs will be useful. This is a guess, but it is not a blind guess. Scan the index and make a list of everything you know that might be relevant. Try them all.*

   Extend the median past the base and lay off an equal length so this double-length segment and the base bisect each other. Connect that point to the endpoints of the base.

2. ***Proof.*** *Using the relevant theorem or construction and any other useful theorems – particularly congruence theorems – prove that the figure has all the information needed to fully define the solution, if only it were begun again with the given lengths and angles.*

   By the parallelogram diagonals theorem, the diagonals of a parallelogram bisect each other, so the solution triangle has the given sides and median.

3. ***Construction****. Begin again with the given lengths and angles and construct the solution triangle using the auxiliary lines or arcs found to be useful. Then erase the auxiliary lines and arcs, which are no longer needed, leaving synthetic knowledge – the solution!*

   By SSS, construct a triangle with the given legs and twice the given median. By SSS, construct a congruent triangle adjacent on the side that is twice the median, and with the equal sides opposite each other so it is a parallelogram, not a kite. The other diagonal cuts this parallelogram into two triangles, both of which are solutions to the problem.

4. ***Discussion.*** *Discuss what conditions the given lengths and angles must meet for a solution to be possible. Also, when a solution is possible, might there be more than one? Are there an infinite number of solutions because the problem is badly posed; that is, under defined?*

   We said, "the other diagonal cuts this parallelogram into two triangles," but, since these triangles are provably congruent, the solution is unique. We have existence so long as the two given sides and twice the given median meet the triangle inequality theorem.

For brevity, this last step is sometimes omitted. Many problems have zero or one solution, depending on whether the triangle inequality theorem is met. Problems involving circles, which we will encounter after proving the inscribed angle theorem, have zero, one or two solutions, depending on whether a line misses, is tangent to or crosses a circle. Students should not neglect this discussion. Real-world problems have non-geometric conditions (*e.g.,* in machine gun emplacement, you will want the solution that is near a rock outcropping), so find them all.

Construction 4.1 *Construct a triangle given its perimeter and two of its angles.*

Solution

Draw $\overline{EFG}$ with the interior angles at vertices E, F, G labeled α, β, γ, respectively. Angles α and β are given but the lengths of the sides are a guess. Find a point J on $\overrightarrow{FE}$ past E such that $\overline{EJ} = \overline{EG}$, and a point K on $\overrightarrow{EF}$ past F such that $\overline{FK} = \overline{FG}$. Thus, $\overline{JK}$ should equal the perimeter, but it does not. How can the figure be redrawn so it does?

By the isosceles angle theorem, $\angle EJG = \frac{\alpha}{2}$ and $\angle FKG = \frac{\beta}{2}$. By ASA (these two angles and the perimeter), construct $\overline{JKG}$. By the mediator theorem, the mediators of $\overline{JG}$ and $\overline{KG}$ intersects $\overline{JK}$ at E and F, respectively. ■

The hypothesis is that the definition of segment is relevant, and so the auxiliary lines needed are to lay off $\overline{EG}$ and $\overline{FG}$ on $\overleftrightarrow{EF}$ so $\overline{JK}$ is the perimeter. Connecting $\overline{JG}$ and $\overline{KG}$ makes two isosceles triangles, so the rest of the construction is basic white-belt geometry. The discussion is that $\overline{JKG}$ is fully defined only if $\frac{\alpha}{2} + \frac{\beta}{2} < \sigma$; otherwise, $\overline{JKG}$ would defy the angle sum theorem.

Construction 4.2 *Construct a triangle given its base, its apex angle and the sum of its legs.*

Solution

Draw $\overline{EFG}$ with the interior angles at vertices E, F, G labeled α, β, γ, respectively. Angle γ and the base $\overline{EF}$ are given, but the lengths of the legs are a guess. Find a point J on $\overrightarrow{FG}$ such that $\overline{EG} = \overline{JG}$. Thus, $\overline{FJ}$ should equal the sum of the legs, but it does not. How can the figure be redrawn so it does? By the isosceles angle theorem, $\angle EJG = \frac{\gamma}{2}$. By ASS, construct $\overline{EFJ}$. By the mediator theorem, the mediator of $\overline{EJ}$ intersects $\overline{FJ}$ at G. ■

Theorems require triangles to be fully defined, but constructions just need a solution, so we can use ASS. $\overrightarrow{JE}$ intersects the F-circle of radius $\overline{EF}$ zero, one or two times; any intersection works.

Thales' Diameter Theorem ***(Euclid, Book III, Prop. 31)***

A chord ***subtends*** *a right angle if and only if it is a diameter.*

Proof

Assume it is a diameter. Connect the vertex of the **inscribed** angle to the center to make two triangles. By the isosceles triangle theorem, their base angles are equal; call them α and β. By the angle sum theorem, $\alpha + (\alpha + \beta) + \beta$ is straight and thus $\alpha + \beta$ is right.

Assume the subtended angle is right. Construct a parallelogram with the chord as the definitional diagonal. By the Lambert theorem corollary, it is a right rectangle. By SAS, the diagonals of a rectangle are equal. By the parallelogram diagonals theorem, the diagonals bisect each other, so the vertices are equidistant from the bi-medial point. Thus, the vertices are **concyclic** and chords through the center are diameters. ■

Thales' Diameter Theorem Corollaries

1. *The circumcenter is inside/outside a triangle if and only if the triangle is acute/obtuse.*
2. *A kite is right if and only if it is cyclic.*

Problem 4.3 *Given a cyclic quadrilateral with sides* $25, 39, 52, 60$ *long, find the circumdiameter.*

Eight–Point Circle Theorem

A quadrilateral $\overline{EFGH}$ *with bi-medial* T *is orthodiagonal iff (1) the midpoints of its sides and the feet of its* ***maltitudes*** *are concyclic; or (2) the feet of perpendiculars dropped from* T*,* $T_{EF}, T_{FG}, T_{GH}, T_{HE}$*, and* $T''_{EF} := \overrightarrow{T_{EF}T} \cap \overline{GH}$ *and* $T''_{FG} := \overrightarrow{T_{FG}T} \cap \overline{HE}$ *and* $T''_{GH} := \overrightarrow{T_{GH}T} \cap \overline{EF}$ *and* $T''_{HE} := \overrightarrow{T_{HE}T} \cap \overline{FG}$ *are concyclic. The (1) and (2) circles coincide iff* $\overline{EFGH}$ *is cyclic.*

Inscribed Angle Theorem ***(Euclid, Book III, Prop. 20, 21, 26, 27)***

1. *Two chords that share an endpoint make an angle half the central angle of their arc.*
2. *Angles with vertices on a circle on the same side of a chord and subtended by it are equal.*
3. *Chords that subtend equal angles inscribed in the same or equal circles are equal.*

Part One, Case One

Two chords with the circle center between them.

Proof

Let $\overline{EF}$ and $\overline{FG}$ be the two chords, O the circle center and F'' the intersection of $\overrightarrow{FO}$ with the circle. Because all radii are equal, $\overline{FOE}$ and $\overline{FOG}$ are both isosceles. By the exterior angle theorem, $\angle EOG = \angle F''OE + \angle F''OG = \angle OFE + \angle OEF + \angle OFG + \angle OGF$. By the isosceles triangle theorem, $\angle EOG = 2\angle OFE + 2\angle OFG = 2\angle EFG$. •

The other part-one cases are the center being on a chord or outside them; proofs are left as exercises. Also left as exercises are parts two and three. Sometimes Thales' diameter theorem is treated as a special case, but it can be proven independently, and I think it should be.

Problem 4.4 *Given* $\overline{EFG}$*, let* E', F' *be the feet of altitudes from* E, F*; and* E'', F'' *be the intersection of* $\overrightarrow{EE'}, \overrightarrow{FF'}$ *with the circumcircle, respectively. Prove that* $\overline{E''G} = \overline{F''G}$*.*

Triangle and Parallelogram Theorem

Given $\overline{EFG}$ and parallelogram $\overline{EJLK}$ with J inside $\overline{EF}$, K inside $\overline{EG}$ and L long of $\angle E$ (past $\overline{FG}$), let $M := \overline{FG} \cap \overline{JL}$ and $N := \overline{FG} \cap \overline{KL}$. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the circumcircles of $\overline{EFG}$, $\overline{JFM}$, $\overline{LNM}$, $\overline{KNG}$, with centers O_1, O_2, O_3, O_4, respectively.

1. $\overline{EFG} \sim \overline{JFM} \sim \overline{LNM} \sim \overline{KNG}$
2. $\overleftrightarrow{EO_1} \parallel \overleftrightarrow{JO_2} \parallel \overleftrightarrow{LO_3} \parallel \overleftrightarrow{KO_4}$
3. *ω_1, ω_2 touch at F. ω_2, ω_3 touch at M. ω_3, ω_4 touch at N. ω_1, ω_4 touch at G.*
4. *$\overline{O_1O_2O_3O_4}$ is a parallelogram.*
5. *Let $\omega_1, \omega_2, \omega_3, \omega_4$ be incircles, not circumcircles; $\overline{O_1O_2O_3O_4}$ is a parallelogram.*

Proof

1. By reflexivity and the transversal theorem, $\overline{EFG} \sim \overline{JFM} \sim \overline{KNG}$, by AA similarity. By the vertical angles theorem, $\overline{LNM}$ also has two of these angles. Note the order of the angles.

This was in the orange belt exit exam; did everybody get it?

2. All corresponding side extensions are parallel, either by collinearity or the parallelogram theorem. If the side extensions are parallel, then $\overleftrightarrow{EO_1} \parallel \overleftrightarrow{JO_2} \parallel \overleftrightarrow{LO_3} \parallel \overleftrightarrow{KO_4}$.

3. By (1), $\overline{EFG} \sim \overline{JFM}$ so, since J is inside $\overline{EF}$, O_2 is inside $\overline{O_1F}$. By the common point theorem, ω_1 and ω_2 touch at F. Analogously for the other three pairs of circles.

4. By (1), $\angle GEF = \angle MJF = \angle MLN = \angle GKN$, which we will call α. By the inscribed angle theorem, $\angle GO_1F = \angle MO_2F = \angle MO_3N = \angle GO_4N = 2\alpha$. Thus, one pair of opposite interior angles in $\overline{O_1O_2O_3O_4}$ is 2α and the other pair is the supplement of 2α. $\overline{O_1O_2O_3O_4}$ is a parallelogram by the parallelogram angles theorem.

The preceding proofs all use the same figure, but the next one must be redrawn with incircles.

5. O_1, O_2 are collinear with F and O_2, O_3 are collinear with M and O_3, O_4 are collinear with N and O_1, O_4 are collinear with G, because they are on the bisectors of angles with these vertices. Let $P := \overline{EF} \cap \overrightarrow{O_4O_1}$ and $Q := \overline{EF} \cap \overrightarrow{O_3O_2}$. By the exterior angle theorem, $\angle EPG = \frac{\angle G}{2} + \angle F$ and $\angle EQM = \frac{\angle M}{2} + \angle F$. But $\angle G = \angle M$ because, by (1), $\overline{EFG} \sim \overline{JFM}$, so $\angle EPG = \angle EQM$ and, by the transversal lemma, $\overleftrightarrow{O_4O_1} \parallel \overleftrightarrow{O_3O_2}$. By the transversal theorem, $\angle KNG = \angle F$ and $\frac{\angle KNG}{2} = \frac{\angle F}{2}$. By the pairwise parallels theorem, $\overleftrightarrow{O_2O_1} \parallel \overleftrightarrow{O_3O_4}$. Thus, $\overline{O_1O_2O_3O_4}$ is a parallelogram by the parallelogram theorem. ■

Problem 4.5 *Find the locus of vertices for a given angle subtended by a given chord.*

Solution

Construct the mediator of the chord. Replicate the given angle on the side long of the proposed vertex with its vertex at one endpoint, then raise a perpendicular to this ray from its endpoint. It intersects the mediator at the center of an arc that passes through the endpoints of the chord. On the other side of the chord, the arc is the locus. ■

Construction 4.3 *Construct a triangle given the apex angle, base altitude and base median.*

Like C. 3.6 but use P. 4.5 on $2m_G$ for the angle of $\sigma - \angle G$. E is the intersection of this arc and the tangent from the midpoint of $2m_G$ to a radius h_G circle around G. Double $\overline{EM_{EF}}$ to find F.

If the endpoints of the chord are two lighthouses, then this locus are the possible positions of a ship where, with a **sextant**, they have measured the subtended angle. Three lighthouses fully define a ship's position. Arcs intersect at two points; one is the middle lighthouse, the other the ship. When we say measure an angle, we are assigning a number of degrees to it only for the purpose of replicating it on the map. Our axioms do not support multiplying these numbers.

Problem 4.6 *You are the captain of a ship out of London and bound for Dublin. As you approach the Cornwall peninsula clockwise, you become disoriented in heavy fog. The lookout reports that he can see three lighthouses, one dead ahead. With a sextant, he measures the angle between the port and bow lighthouses as* $80°$*, and the angle between the starboard and bow lighthouses as* $120°$*. Locate your ship on the map.*

Because light travels at a constant speed, about three meters in a nanosecond, a laser rangefinder with a clock this accurate can measure the time it takes for light to go out to an object and be reflected back. A GPS unit is a receiver that gets signals from satellites with in-sync clocks. If the GPS unit's clock were in-sync, location would be the intersection point of three spheres. It is too expensive to keep the GPS unit's clock in sync, so it requires more math and line of sight on at least four satellites. But no angles are ever measured, so it is not like a sextant.

Isaac Newton used geometry to invent the sextant, though it was not until about 1730 when the Englishman John Hadley and the American Thomas Godfrey independently began manufacturing the devices. The name implies that they can only measure angles up to φ, a sixth of a circle [86], though we will here assume that they can measure angles up to σ. Today, laser rangefinders no larger than a pair of opera glasses allow any soldier or outdoorsman to measure the distances to objects a kilometer away; the best ones can measure the distance to the moon. These did not exist until the 21st century. Sextants remain very delicate optical instruments that are kept on board ships in padded and watertight cases. Thus, there are two applications for triangle construction problems. In sextant problems, one is given the base, the apex angle and one other piece of information about the triangle. In laser problems, one is given the lengths of three segments in a triangle; *e.g.,* a side, a median, an altitude, its inradius or its circumradius. Roughly speaking, these are the problems motivated by marine and land navigation, before GPS, and what we will do again if we fight a real army (not hillbillies in Afghanistan) that can down satellites.

The following table classifies the more difficult constructions of triangle $\overline{EFG}$ in which $\overline{EF}$ is the base, m_E, m_F, m_G the medians, h_E, h_F, h_G the altitudes, t_E, t_F, t_G the angle bisectors, r the inradius, R the circumradius and s the semiperimeter. x_G is either m_G, h_G or t_G.

Sextant Problems		**Laser Problems**		**Other Triangle Constructions**	
C. 4.2	$\overline{EF},\ \angle G,\ \overline{FG}+\overline{EG}$	C. 3.6	$\overline{FG},\ \overline{EG},\ m_G$	C. 3.7	$\overline{EF},\ \angle F,\ m_E$
C. 4.6	$\overline{EF},\ \angle G,\ \angle E$	C. 4.11	$\overline{EF},\ R,\ m_E$	C. 4.1	$2s,\ \angle E,\ \angle F$
C. 4.7	$\overline{EF},\ \angle G,\ \overline{FG}-\overline{EG}$	C. 4.13	$\overline{EF},\ R,\ r$	C. 4.3	$h_G,\ m_G,\ \angle G$
C. 4.8	$\overline{EF},\ \angle G,\ h_E+h_F$	C. 4.14	$m_E,\ m_F,\ m_G$	C. 4.12	$R,\ r,\ \angle G$
C. 4.9	$\overline{EF},\ \angle G,\ h_E-h_F$	C. 4.15	$m_E,\ m_F,\ h_G$	C. 4.18-19	$R, \overline{FG}+\overline{EG}, \angle E \mp \angle F$
C. 4.10	$\overline{EF},\ \angle G,\ h_G$	C. 4.20	$m_G,\ h_G,\ t_G$	C. 4.21	$R,\ \angle E-\angle F,\ x_G$
		C. 4.22	$\overline{EF},\ h_G,\ t_G$	C. 5.1	Equilateral; J, K, L on sides

Brahmagupta's Bi–Medial Theorem

Given $\overline{EFGH}$ cyclic with $\overline{EG} \perp \overline{FH}$ at T, if T' is the foot of the perpendicular dropped on $\overline{EF}$ from T, then $M := \overrightarrow{T'T} \cap \overline{GH}$ is the midpoint of $\overline{GH}$; that is, $M \equiv M_{GH}$.

Proof

By the inscribed angle theorem, $\angle EFH = \angle EGH$, which we will call α. $\angle EFH = \angle ETT'$ by the pairwise perpendiculars theorem, and $\angle ETT' = \angle GTM$ by the vertical angles theorem. Thus, by the isosceles triangle theorem converse, $\overline{TM} = \overline{GM}$. By the inscribed angle theorem, $\angle FEG = \angle FHG$, which is complementary to α in the right triangle $\overline{T'ET}$. $\angle GTH$ is given right, so $\angle GTM$ is complementary to $\angle HTM$ and $\angle HTM = \angle THM$. By the isosceles triangle theorem converse, $\overline{TM} = \overline{HM}$. Thus, $\overline{GM} = \overline{HM}$; so, $M \equiv M_{GH}$. ■

[86] Calling it the hextant would have saved generations of teenage boys from fits of giggles in their geometry class.

A maltitude (midpoint-altitude) is the perpendicular dropped from the midpoint of a side of a quadrilateral onto the opposite side.[87] In the previous theorem, we learned that, if the quadrilateral is both cyclic and orthodiagonal (its diagonals are perpendicular), then the maltitudes are concurrent at its bi-medial; that is, at the intersection of its diagonals, T. The maltitudes' point of concurrency, if it exists, is called the **anticenter**, denoted S unless it is T.

Anticenter Theorem

1. *A quadrilateral is cyclic if and only if the maltitudes are concurrent.*
2. *The medial point is midway between the circumcenter and the anticenter.*

The medial point, C, is the intersection of the bimedians of a quadrilateral; it is also where the medians of a triangle are concurrent. The context should make it clear which meaning is in use.

Proof

Assume that $\overline{EFGH}$ is cyclic with O the circumcenter, C the medial point and S a point on $\overrightarrow{OC}$ such that $\overline{OC} = \overline{CS}$. By the Varignon theorem, the bimedians bisect each other so, by SAS, $\overline{M_{HE}CS} \cong \overline{M_{FG}CO}$. Thus, $\angle M_{HE}SO = \angle M_{FG}OS$ and, by the transversal lemma, $\overleftrightarrow{M_{HE}S} \parallel \overleftrightarrow{OM_{FG}}$. By the diameter and chord theorem, $\overleftrightarrow{OM_{FG}} \perp \overline{FG}$, and by parallelism, $\overleftrightarrow{M_{HE}S} \perp \overline{FG}$. Analogously, the other three maltitudes also all pass through S.

Assume that the maltitudes are concurrent at S, so any one of the lines through S and the midpoint of a side is perpendicular to the opposite side. In particular, $\overleftrightarrow{M_{HE}S} \perp \overline{FG}$; just walk backwards through the proof above to get $\overline{OC} = \overline{CS}$, collinear. ■

If cyclic $\overline{EFGH}$ is orthodiagonal, then its anticenter and its bi-medial coincide; and if it is not?

Lemma 4.1

The bimedians of $\overline{EFGH}$ intersect at the bi-medials of $\overline{M_{EF}M_{EG}M_{GH}M_{FH}}$ and $\overline{M_{FG}M_{FH}M_{HE}M_{EG}}$.

Look to medial parallelogram theorem II and to the parallelogram diagonals theorem.

Anticenter–Orthocenter Theorem

Given $\overline{EFGH}$ cyclic with T its bi-medial and S its anticenter, S is the orthocenter of $\overline{M_{EG}M_{FH}T}$.

Prove that $\overline{M_{EG}SM_{FH}O}$ is a parallelogram, so $\overline{OC} = \overline{CS}$; then cite the anticenter theorem.

[87] Alternate usage: When I take my entire geometry class out for malts, the soda jerk exclaims, "It's a maltitude!"

Problem 4.7 *Prove that, in a cyclic and orthodiagonal quadrilateral, the distance from the circumcenter to a side is half the opposite side.*

Construction 4.4 *Through a point outside a circle, draw a line tangent to the circle.*

> ***Green Belt Solution***
> Connect the point with the center and find the midpoint of this segment. By Thales' diameter theorem, from any point on the circle around this diameter, the angle made by lines to the endpoints is right; this includes the two points where it intersects the given circle. By the tangent theorem, these are the two touching points. ■

Construction 4.5 *Given the hypotenuse and a leg of a right triangle, construct the other leg.*

C. 4.4 is easier to draw than C. 2.2 because the auxiliary circle is half as large. Construction 2.2 allows first-year geometers to construct tangents, which gives them a shot at the IMO. But now that we have C. 4.4, we should just put C. 2.2 into the past. C. 4.5 is the same construction but stated without reference to a circle.

Construction 4.6 *Construct a triangle given its base, its apex angle and a base angle.*

Hypothesis: Since we have the base and the apex angle, P. 4.5, finding the locus of vertices for a given angle subtended by a given chord, is probably relevant.

Proof: State that any point on the constructed arc has the given base and apex angle.

Construction: By P. 4.5, find the locus of vertices for the apex angle, α, subtended by the base. By C. 1.5, construct the base angle, β, at one end of the base. This ray and the arc of possible vertices intersect at the apex. Connect the other leg of the triangle.

Discussion: If $\alpha + \beta > \sigma$, the angle sum theorem is violated, and the loci do not intersect. If they do, the given base angle could have its vertex at either end of the base, so there are two triangles, but they are provably congruent, so the solution is unique.

Construction 4.7 *Construct a triangle given its base, its apex angle and the difference of its legs.*

Guess at the solution by drawing the desired triangle, $\overline{EFG}$, given $\overline{EF}$, $\angle G$ and J on the longer leg, $\overline{FG}$, so $\overline{EJG}$ is isosceles. Hypothesis: the mediator theorem will locate G if $\overline{EFJ}$ can be constructed and the isosceles angle theorem applied to $\overline{EJG}$ may be helpful in constructing $\overline{EFJ}$.

In construction 4.6, our hypothesis was, "Since we have the base and the apex angle, P. 4.5, finding the locus of vertices for a given angle subtended by a given chord, is probably relevant." Indeed, it was relevant! But, in the following problem, it is not relevant, which should serve as a reminder of the aphorism, "If at first you do not succeed, then try, try again taking another path."

Construction 4.8 *Construct a triangle given its base, its apex angle and the sum of the altitudes to the legs.*

Hypothesis: Guess at the solution by drawing the desired triangle with the given base and altitudes to the legs that have the given sum, but the apex angle only approximate. Extend one altitude past its leg and lay off the length of the other altitude. Orange belts were often given a length somewhere that it was not useful, but they then used the Lambert theorem to construct it on the opposite side of a right rectangle where it *was* useful. Our hypothesis is that this is actually an orange-belt problem.

Proof: Given $\overline{EFG}$ with E' and F' the feet of altitudes, lay off $\overline{FF'}$ on $\overrightarrow{EE'}$ past E' to J and draw a rectangle, $\overline{GE'JK}$. Let $F'' := \overleftrightarrow{JK} \cap \overrightarrow{EF}$ and $G'' := \overleftrightarrow{JK} \cap \overrightarrow{EG}$. By T&V, $\angle EGF = \angle EG''F''$, so, by AAS, $\overline{GF'F} \cong \overline{G''KG}$; thus, $\overline{GF} = \overline{G''G}$. By the isosceles angle theorem, $\angle GG''F = \frac{1}{2}(\sigma - \angle FGG'') = \frac{1}{2}\angle EGF = \frac{1}{2}\angle EG''F''$. Thus, $\overrightarrow{G''F}$ bisects $\angle EG''F''$. By the mediator theorem, G is on the mediator of $\overline{G''F}$.

Construction: Draw parallel lines of the given width $\overline{EE'} + \overline{FF'}$. Taking one as $\overleftrightarrow{G''F''}$, lay the given apex angle against it so the vertex is G'' and the ray and the other line intersect at E. Bisect $\angle EG''F''$. A circle around E whose radius is the given base intersects this bisector at F. G is the intersection of the mediator of $\overline{G''F}$ and $\overrightarrow{G''E}$.

Discussion: "A circle around E whose radius is the given base intersects this bisector at F." A circle and a ray intersect at zero, one or two points; this is the number of solutions.

Construction 4.9 *Construct a triangle given its base, its apex angle and the difference of the altitudes to the legs.*

This is essentially the same problem as C. 4.8 except, where before we said, "extend one altitude past its leg and lay off the length of the other altitude," we now lay it off in the other direction, so $\overline{EJ}$ is the difference of the altitudes to the legs, not their sum. Thus, our rectangle is inside the triangle, not sticking out from it. The construction is left as an exercise; if you cannot do C. 4.9 on your own, then you must not have really understood C. 4.8 when it was explained to you.

Construction 4.10 *Construct a triangle given its base, its apex angle and the altitude to its base.*

Solution

By P. 4.5, find the locus of vertices for the apex angle subtended by the base. Construct a line parallel to the base and at a distance from it equal to the given altitude on that side of the base. Where it intersects the locus of vertices is the apex. A line intersects an arc at zero, one or two points; this is how many solutions there are on a side of the base. ■

Problem 4.8 *Find the locus of the midpoints of chords in a given circle passing through a given point on or inside the circle.*

Solution

Let O be the circle center and P be the point. The locus is the circle with diameter $\overline{OP}$. ■

Proof

By Thales' diameter theorem, $\angle OMP$ is right for any M on the locus circle. Since $\overline{OM}$ passes through the center of the given circle, it can be extended to be a diameter. By the diameter and chord theorem, M is the midpoint of the chord through P. ■

Construction 4.11 *Construct a triangle given its base, its* ***circumradius****, and the median to its base or to a leg.*

If the median is to the base, the construction is easy; if it is to a side, then one of the loci is the solution to P. 4.8. Either way, the apex is on two arcs, which intersect at zero, one or two places.

Construction 4.12 *Construct a triangle given its inradius, circumradius and an interior angle.*

Solution

Draw the circumcircle, ω, and label a point on it G''. At this vertex, replicate the given angle inside the circumcircle so it cuts off the chord $\overline{EF}$. Draw a parallel to $\overleftrightarrow{EF}$ on the G'' side of width r, the inradius. By P. 4.5, find the locus of vertices for $\rho + \frac{1}{2}\angle G$ subtended by $\overline{EF}$ and on the G'' side. Where this arc intersects the parallel to $\overleftrightarrow{EF}$ is the incenter, I. Find G on ω such that $\angle FEG = 2\angle FEI$. ■

Discussion

There are zero, one or two possible incenters, I. Zero if $R < 2r$, one if $R = 2r$, or two if $R > 2r$. In the latter case, the two triangles are congruent, so the solution is unique. ■

Proof

By the inscribed angle theorem, $\overline{EF}$ subtends the given angle for any G on the G'' side of the circumcircle. We need G so $\overline{EFG}$ has the given inradius; suppose we have found it.

1. $\angle E + \angle F + \angle G = \sigma$	Angle sum theorem for $\overline{EFG}$
2. $\angle IEF + \angle IFE + \frac{1}{2}\angle G = \rho$	Bisect all the angles in (1)
3. $\angle IEF + \angle IFE + \angle EIF = \sigma$	Angle sum theorem for $\overline{EFI}$
4. $\rho - \frac{1}{2}\angle G = \sigma - \angle EIF$	Solve (2) and (3) for $\angle IEF + \angle IFE$ and set equal.
5. $\angle EIF = \rho + \frac{1}{2}\angle G$	Solve for $\angle EIF$ ■

Construction 4.13 *Construct a triangle given its inradius, circumradius and a side.*

This is the same as C. 4.12, but you start with $\overline{EF}$ and do not need G''.

Problem 4.9 *Find lengths e and g such that $e + g = z$ and $e^2 + g^2 = f^2$ with z and f given.*

In America, this would be an Algebra II problem because, while Algebra I students have the quadratic equation, solving the equations simultaneously, collecting like terms and applying the quadratic formula is too much for them. But a geometer would see that, by Thales' diameter theorem, e and g are the legs of a right triangle with hypotenuse f, which is its circumdiameter. With the right triangle incircle theorem and the triangle centers' angles theorem, it is easy!

Solution

By the right triangle incircle theorem, the indiameter is $d = z - f$. $\overline{EFG}$ is right, so f is the diameter of its circumcircle, ω, by Thales' diameter theorem. Draw a parallel to the diameter, l, at width $r = \frac{1}{2}d = \frac{1}{2}(z - f)$. Draw the mediator to f and, from where it intersects ω on the other side of l, draw an arc through the endpoints of f. By the triangle centers' angles theorem part two and problem 4.5, it cuts l at the incenter. The incenter is on the angle bisector, so we have a ray and a point on the angle bisector; by C. 1.6, find the other ray of the angle. This ray cuts ω in one triangle leg. Connect the other leg. ■

Discussion

The arc around where the mediator intersects the circle intersects the line on the other side of the diameter, parallel to it and at width r; it will at zero, one or two points. ■

In addition to being easier than algebra, geometry clarifies the discussion. Just a glance tells you that the arc radius is $\sqrt{2}R$; thus, a solution exists if $R + \frac{z-f}{2} \leq \sqrt{2}R$ or $z \leq \sqrt{2}f$. Easy!

For example, suppose that $z = 8''$ and $f = 6''$ and I want e and g to the nearest 32nd of an inch. It takes only a few minutes to do the geometric construction and then measure the triangle sides with a ruler to get $2\frac{19}{32}''$ and $5\frac{13}{32}''$. Working through all the algebra to get $e = \frac{z \pm \sqrt{2f^2 - z^2}}{2}$ and then evaluating it on a slide rule would take longer and result in $2.6''$ and $5.4''$, which is only accurate to a 10th of an inch. Of course, we now have calculators, which are quicker and more accurate. But we are here to learn logic, not to plug numbers into formulas. No employer wants a new hire with a diploma, a calculator, and a hazy memory of formulas. And no history teacher wants to see Americans' logic so weak that we elect a pathological liar to be our president.

When you became a green belt, you became an intermediate geometer, which means that you are expected to help the white- and yellow-belt students. Recall from the white-belt chapter,

> ***Teachers!*** *If you have read this far hoping for advice on how to get your #%$^@ students through the* Common Core *standardized exam, here it is: Ask for the perimeter of a triangle with vertices* $(-2,3), (-4,-4), (-7,-1)$ *and make it a race. The easy way is to lay the three sides end-to-end on a line. Taking the sum of three applications of the algebraic distance formula is the hard way.*
>
> $$\sqrt{(-2-(-7))^2+(3-(-1))^2}+\sqrt{(-2-(-4))^2+(3-(-4))^2}+\sqrt{(-7-(-4))^2+(-1-(-4))^2} \approx 17.9$$

It should be clear by now why this is so important. Algebra makes geometry needlessly difficult!

Problem 4.10

Given $\overline{EFG}$ *and* $\overleftrightarrow{GS} \parallel \overleftrightarrow{EF}$ *and* $J := \overleftrightarrow{M_{EF}M_{FG}} \cap \overleftrightarrow{GS}$, *prove that* $\overline{EM_{FG}} = \overline{M_{GE}J}$ *and* $\overline{M_{EF}G} = \overline{JF}$.

> ***Solution***
>
> By mid-segment theorem #1, $\overline{M_{GE}M_{FG}} = \frac{1}{2}\overline{EF} = \overline{EM_{EF}}$ and $\overleftrightarrow{M_{GE}M_{FG}} \parallel \overleftrightarrow{EM_{EF}}$. By the equal segments on parallels theorem, $\overline{EM_{EF}M_{FG}M_{GE}}$ is a parallelogram. By T&V, $\angle M_{GE}EM_{EF} = \angle GM_{GE}M_{FG}$ so $\overline{M_{GE}EM_{EF}} \cong \overline{GM_{GE}M_{FG}}$. $\overline{EM_{EF}M_{FG}M_{GE}} \cong \overline{M_{GE}M_{FG}JG}$ because their definitional triangles are congruent. Thus, their diagonals $\overline{EM_{FG}} = \overline{M_{GE}J}$. $\overline{EM_{EF}M_{FG}M_{GE}} \cong \overline{M_{GE}M_{FG}JG}$ also implies $\overline{EM_{EF}} = \overline{GJ}$. But $\overline{EM_{EF}} = \overline{M_{EF}F}$ so, by the equal segments on parallels theorem, $\overline{M_{EF}FJG}$ is a parallelogram and $\overline{M_{EF}G} = \overline{JF}$. ■

Now let us see if we can do it in reverse! Instead of being given the triangle and asked to prove something about the medians, let us be given the medians and try to construct the triangle. In P. 4.10, we prove that $\overline{FJGM_{EF}}$ is a parallelogram, which holds the equalities we need. In C. 4.14, we construct the same parallelogram, $\overline{EJCK} \cup \overline{JFLC}$, by laying off some equal segments.

Construction 4.14 *Construct a triangle given the lengths of the three medians.*

Solution

By SSS, construct a triangle with the three medians as its sides and label the vertices E, F, G. By C. 3.11, trisect $\overline{EG}$ and label as C the point that is twice as far from E as from G. By C. 3.3, draw a line through C parallel to $\overleftrightarrow{EF}$. By C. 3.11, trisect $\overline{EF}$ and label as J the point that is twice as far from F as from E. By the equal segments on parallels theorem, find K, L so $\overline{EJCK}, \overline{JFLC}$ are parallelograms. Let $P := \overrightarrow{EK} \cap \overrightarrow{LG}$. $\overline{ELP}$ is the triangle. ■

Discussion

The three given medians must satisfy the triangle inequality theorem. ■

Side–Angle–Side (SAS) Third–Scale Triangle Theorem

If a triangle has two sides that are a third the corresponding sides in another triangle and the included angles are equal, then the other angles are equal and the other side also a third.

Angle–Side–Angle (ASA) Third–Scale Triangle Theorem

If two pairs of angles are equal in two triangles and the included side of one triangle is a third the included side in the other triangle, then the other sides are also a third their corresponding sides.

Green belts can use the easy ratios $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$. Similarity will allow us to use any ratio $\frac{m}{n}$. Indeed, most "similarity" problems in other textbooks are these easy ratios, so you have got that.

Lemma 4.2

A triangle's medial point is a third of the way from the base to the apex.

Proof

By T&V and the two-to-one medial point and ASA third-scale triangle theorems. ■

Construction 4.15 *Construct a triangle given the lengths of two medians and the altitude to the other side.*

Solution

By C. 3.11, trisect all three of the given lengths. Draw parallel lines a third the altitude apart; by lemma 4.2, if the base is on one line, the medial point is on the other. Put C on one line and draw arcs of two thirds the given medians to intersect the other line at E and F. $\overline{EF}$ will be the base. Extend $\overrightarrow{EC}$ and $\overrightarrow{FC}$ by half again to E'' and F'', respectively. $G := \overrightarrow{EF''} \cap \overrightarrow{FE''}$ is the apex. ■

Discussion

If two-thirds of the shorter median is less than one-third of the altitude, there is no solution. If it is equal, then $\overline{CEF}$ is a right triangle and there is one solution. If it is greater, then $\overline{CEF}$ can be either an acute or an obtuse triangle and there are two solutions. ■

Tangent and Chord Theorem ***(Euclid, Book III, Prop. 32)***

A line intersects a circle where it makes an angle with a chord equal to the angle subtended by that chord if and only if that is a touching point.

Proof

Assume $\overleftrightarrow{EJ}$ is tangent to the circumcircle (center O) of $\overline{EFG}$ at E, J long of $\angle G$. By the tangent theorem, $\angle OEJ$ is right and, by the diameter and chord theorem, $\angle OM_{EF}E$ is right. $\angle JEF = \angle EOM_{EF}$, both complementary to $\angle OEM_{EF}$. By SSS, $\overline{EOM_{EF}} \cong \overline{FOM_{EF}}$, so $\angle EOM_{EF} = \angle FOM_{EF}$ and $\angle EOF = 2\angle EOM_{EF}$. By the inscribed angle theorem, $\angle EOF = 2\angle G$. Thus, $\angle EOM_{EF} = \angle G$ and $\angle JEF = \angle G$. •

Proof of the converse is left as an exercise; just walk backwards through the proof.

Construction 4.16 ***(Euclid, Book IV, Prop. 2)***

Given a circle and a triangle, inscribe a similar triangle in the circle.

Solution

Let α, β, γ be the angles of the given triangle and ω be the circle; if the triangle is right, then label that angle γ. By C. 3.1, locate the circle center, O, if it is not already known. Choose a point G on ω for the γ vertex and, by C. 1.3, draw a line at G perpendicular to $\overleftrightarrow{OG}$ with J and K on this line and on opposite sides of $\overleftrightarrow{OG}$. By C. 1.5, find E on ω so $\angle EGJ = \beta$ and F on ω so $\angle FGK = \alpha$. By the tangent and chord theorem, $\angle EFG = \beta$ and $\angle FEG = \alpha$. By AA similarity, $\overline{EFG}$ is the desired triangle. ■

Discussion

The γ vertex is wherever G is and the α vertex is on the same side of $\overleftrightarrow{OG}$ as J. ■

Intersecting Secant and Tangent Similarity Theorem

If P is the intersection of $\overrightarrow{GF}$ and the tangent to the circumcircle of $\overline{EFG}$ at E, then $\overline{PEF} \sim \overline{PGE}$.

Proof

Switch F and G if needed. By the tangent and chord theorem, $\angle PEF = \angle PGE$. $\overline{PEF}$ and $\overline{PGE}$ both contain $\angle P$, which is two equal angles; thus, $\overline{PEF} \sim \overline{PGE}$ by AA similarity. ■

Intersecting Secants Similarity Theorem

Given $\overline{EFGH}$ cyclic, assume $P := \overrightarrow{FE} \cap \overrightarrow{GH}$ exists; then, (1) $\overline{PFH} \sim \overline{PGE}$, and (2) $\overline{PFG} \sim \overline{PHE}$.

Proof

1. $\angle PFH = \angle PGE$ by the inscribed angle theorem; with $\angle P$, $\overline{PFH} \sim \overline{PGE}$ by AA similarity.
2. By the angle sum theorem in $\overline{EFG}$, $\angle EFG + \angle FGE + \angle GEF = \sigma$; by supplementarity, $\angle PHE + \angle FHE + \angle GHF = \sigma$ or $\angle PHE + \angle FGE + \angle GEF = \sigma$ by the inscribed angle theorem. Thus, $\angle PHE = \angle PFG$. With $\angle P$, $\overline{PFG} \sim \overline{PHE}$ by AA similarity. ■

Intersecting Chords Similarity Theorem

Given $\overline{EFGH}$ cyclic and T its bi-medial, then (1) $\overline{EFT} \sim \overline{HGT}$, and (2) $\overline{FGT} \sim \overline{EHT}$.

Proof

1. Let $\overline{EFGH}$ be cyclic and T be its bi-medial. $\angle ETF = \angle GTH$ by the vertical angles theorem. $\angle GEF = \angle GHF$ and $\angle EGH = \angle EFH$ by the inscribed angle theorem. Thus, $\overline{EFT} \sim \overline{HGT}$.
2. Analogously, $\overline{FGT} \sim \overline{EHT}$. ■

The next two theorems lend themselves to the computerized exams promoted by Bill Gates.

Intersecting Chords Angle Theorem

The angle made by intersecting chords is the semisum of the two arcs they cut off.

Proof

Given $\overline{EFGH}$ cyclic and T its bi-medial, let angle α be subtended at the center by $\overline{EF}$ and angle β be subtended at the center by $\overline{GH}$ and $\gamma = \angle ETF = \angle GTH$, which are equal by the vertical angles theorem. By the external angle theorem, $\gamma = \angle FHE + \angle GEH$. By the inscribed angle theorem, $\angle FHE = \frac{\alpha}{2}$ and $\angle GEH = \frac{\beta}{2}$ and so $\gamma = \frac{\alpha}{2} + \frac{\beta}{2} = \frac{\alpha + \beta}{2}$. ■

Intersecting Secants Angle Theorem

The angle made by intersecting secants is the semidifference of the far and near arc.

Proof

Given $\overline{EFGH}$ cyclic, assume $P := \overrightarrow{GF} \cap \overrightarrow{HE}$ exists (reorder the vertices if it does not). Let α be the angle subtended at the center by $\overline{EF}$ and β be the angle subtended at the center by $\overline{GH}$ and $\gamma = \angle EPF$. By the exterior angle theorem, $\gamma = \angle GFH - \angle EHF$. $\angle GFH = \frac{\beta}{2}$ and $\angle EHF = \frac{\alpha}{2}$ by the inscribed angle theorem, so $\gamma = \frac{\beta}{2} - \frac{\alpha}{2} = \frac{\beta - \alpha}{2}$. ■

Cyclic Quadrilateral Theorem ***(Euclid, Book III, Prop. 22)***
If a quadrilateral is cyclic, then its opposite angles are supplementary.

Proof
Let $\overline{EFGH}$ be a cyclic quadrilateral. By the inscribed angle theorem, $\angle EGH = \angle EFH$ and $\angle EGF = \angle EHF$. By substitution, $\angle FGH = \angle EGH + \angle EGF = \angle EFH + \angle EHF$. By the angle sum theorem, $\angle EFH + \angle EHF + \angle HEF = \sigma$. Thus, $\angle FGH + \angle HEF = \sigma$. ■

Cyclic Quadrilateral Theorem Corollary
Given $\overline{EFG}$, the circumcircles of exterior triangles $\overline{E''FG}$, $\overline{EF''G}$, $\overline{EFG''}$ are concurrent if and only if $\angle E'' + \angle F'' + \angle G'' = \sigma$.

Cyclic Quadrilateral Theorem Converse
If a quadrilateral has two opposite angles that are supplementary, then it is cyclic.

Proof
Given $\overline{EFGH}$ with $\angle H + \angle F = \sigma$, H is either inside, on or outside the $\overline{EFG}$ circumcircle, ω. If H is outside ω, let $J := \overline{EH} \cap \omega$ (switch E and G if $\overline{EH}$ does not cut ω) so $\overline{EFGJ}$ is cyclic and, by the cyclic quadrilateral theorem, $\angle GJE$ is supplementary to $\angle F$ and thus equal to $\angle H$, which was given to be supplementary to $\angle F$. This contradicts the exterior angle theorem. Analogously, H inside ω is also contradictory; thus, H must be on ω. ■

Right Cyclic Theorem ***(formerly quadrilateral angle sum theorem corollary #1)***
If opposite quadrilateral angles are right, then the other two angles are supplementary.

Construction 4.17 ***(Euclid, Book IV, Prop. 3)***
Given a circle and a triangle, circumscribe a similar triangle around the circle.

Solution
Let α, β, γ be the angles of the given triangle and ω_I be the circle. By C. 3.1, locate the circle center if it is not already known; call it I because it will be the new triangle's incenter. Choose a point I_G on ω_I for the touching point between the α and β vertices and, by C. 1.3, draw a line at I_G perpendicular to $\overleftrightarrow{II_G}$. By C. 1.5, find I_E on ω_I on the same side of $\overleftrightarrow{II_G}$ as where you want the β vertex so $\angle I_E I I_G$ is the supplement of β. Analogously, find I_F on ω_I on the other side of $\overleftrightarrow{II_G}$ so $\angle I_F I I_G$ is the supplement of α. By the right cyclic theorem, the lines tangent at I_F and I_G intersect at a point E such that $\angle E = \alpha$, and the lines tangent at I_E and I_G intersect at a point F such that $\angle F = \beta$. If G is the intersection of the lines tangent at I_E and I_F, then, by AA similarity, $\overline{EFG}$ is the desired triangle. ■

Napoleon Theorem

The centers of equilateral triangles external to triangle sides form an equilateral triangle.

Circumcenter Proof

Given $\overline{EFG}$, let G'', E'', F'' be the apexes of equilateral triangles built on the exteriors of $\overline{EF}, \overline{FG}, \overline{GE}$, respectively. Let O_E, O_F, O_G be the circumcenters of $\overline{E''FG}$, $\overline{EF''G}$, $\overline{EFG''}$, respectively. By the cyclic quadrilateral theorem corollary, they are concurrent at a point, P. By the common chord theorem, their common chords mediate their lines of centers. If $\angle O_E, \angle O_F$ and $\angle O_G$ are the interior angles of $\overline{O_E O_F O_G}$, then $\angle O_E + \angle FPG = \sigma$ and $\angle O_F + \angle GPE = \sigma$ and $\angle O_G + \angle EPF = \sigma$ by the right cyclic theorem. $\angle E'' + \angle FPG = \sigma$ and $\angle F'' + \angle GPE = \sigma$ and $\angle G'' + \angle EPF = \sigma$ by the cyclic quadrilateral theorem. Thus, $\angle O_E = \angle E''$ and $\angle O_F = \angle F''$ and $\angle O_G = \angle G''$, so $\overline{O_E O_F O_G}$ is equilateral. ■

Application of the cyclic quadrilateral theorem corollary is easy because all equilateral triangles are similar, no matter how the vertices are ordered. But any three similar triangles built on the sides of a triangle satisfy this requirement if their apexes differ; that is, $\overline{E''FG} \sim \overline{EF''G} \sim \overline{EFG''}$.

Red belts will learn of the Miquel theorem: Given $\overline{EFG}$ and arbitrary points J, K, L on $\overline{EF}, \overline{FG}, \overline{GE}$, respectively, the circumcircles of $\overline{EJL}, \overline{FKJ}$ and $\overline{GLK}$ are concurrent. E, F, G are on the sides of the triangle of excenters, $\overline{XYZ}$; thus, the Miquel concurrency is a way to make $\overline{EFX} \sim \overline{EYG} \sim \overline{ZFG}$.

Lemma 4.3

If $\overline{EFG} \sim \overline{JKL}$ then $\overline{EM_{EF}G} \sim \overline{JM_{JK}L}$.

Butterfly Theorem

Given $\overline{EFGH}$ cyclic with circumcenter O and bi-medial T, T is the midpoint of a chord perpendicular to $\overleftrightarrow{OT}$; let it intersect $\overline{EF}$ at J and $\overline{GH}$ at K. Then, $\overline{TJ} = \overline{TK}$.

In the classic figure that looks like a butterfly, M_{EF}, M_{GH} and O are all on the same side of $\overleftrightarrow{JK}$; we will prove the theorem for this case. The other cases are analogous and will be left as exercises.

Green Belt Proof

$\overline{EFT} \sim \overline{HGT}$ by the intersecting chords similarity theorem. $\overline{EM_{EF}T} \sim \overline{HM_{GH}T}$ by lemma 4.3 and so $\angle EM_{EF}T = \angle HM_{GH}T$. $\angle EM_{EF}O = \rho = \angle HM_{GH}O$ by the diameter and chord theorem. Thus, $\overline{JM_{EF}OT}$ and $\overline{KM_{GH}OT}$ are cyclic by the cyclic quadrilateral theorem converse. $\angle JM_{EF}T = \angle JOT$ and $\angle KM_{GH}T = \angle KOT$ by the inscribed angle theorem; so, by transitivity, $\angle JOT = \angle KOT$. By ASA, $\overline{OTJ} \cong \overline{OTK}$, so $\overline{TJ} = \overline{TK}$. •

Triangle Frustum Theorem III

Given $\overline{EFGH}$ and its bi-medial T, $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ if and only if the circumcircles of $\overline{EFT}$ and $\overline{GHT}$ touch.

> ***Proof***
>
> Assume that $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$. O_1 and O_2 are the circumcenters of $\overline{EFT}$ and $\overline{GHT}$, respectively. $\angle EFH = \angle FHG = \alpha$ by the transversal theorem; thus, $\angle EO_1T = \angle GO_2T = 2\alpha$ by the inscribed angle theorem. By the isosceles angle theorem, $\angle ETO_1 = \angle GTO_2 = \rho - \alpha$. By the vertical angles theorem, T is collinear with O_1 and O_2. By the common point theorem, the circumcircles of $\overline{EFT}$ and $\overline{GHT}$ touch. •

Proof of the converse is left as an exercise. Note that the vertical angles theorem is biconditional; most textbooks assume collinearity and prove the angles equal, but they neglect the converse.

Next, in spiral similarity, we assume that S and H are on opposite sides of $\overleftrightarrow{EG}$. Proof for S and H on the same side of $\overleftrightarrow{EG}$ is analogous and is left as an exercise.

Spiral Similarity Theorem

Given $\overline{EFGH}$ with bi-medial T and $\overleftrightarrow{EF} \nparallel \overleftrightarrow{GH}$, if S is the other intersection of the circumcircles of $\overline{EFT}$ and $\overline{GHT}$, then $\overline{EFS} \sim \overline{GHS}$.

> ***Proof***
>
> By triangle frustum theorem III, the circumcircles of $\overline{EFT}$ and $\overline{GHT}$ intersect at two points, S and T. $\angle ESF = \angle ETF = \angle HTG = \angle HSG$ by the inscribed angle and vertical angles theorems. By supplementarity, $\angle STG + \angle GTH = \sigma - \angle FTS$. By the cyclic quadrilateral theorem, $\angle STG + \angle GTH = \sigma - \angle SGH$. Thus, $\angle FTS = \angle SGH$. By the inscribed angle theorem, $\angle FES = \angle SGH$. With $\angle ESF = \angle HSG$, by AA similarity, $\overline{EFS} \sim \overline{GHS}$. ■

Spiral Similarity Theorem Converse

Given $\overline{EFGH}$ and S such that $\overline{EFS} \sim \overline{GHS}$, if T is another intersection of the circumcircles of $\overline{EFS}$ and $\overline{GHS}$, then $\overleftrightarrow{EF} \nparallel \overleftrightarrow{GH}$ and T is the bi-medial of $\overline{EFGH}$.

> ***Proof***
>
> By similarity, $\angle FES = \angle HGS$. By the inscribed angle theorem, $\angle FES = \angle FTS$. Thus, $\angle HGS = \angle FTS$. By the cyclic quadrilateral theorem, $\angle STG + \angle GTH = \sigma - \angle HGS$, so $\angle STG + \angle GTH + \angle FTS = \sigma$ and F, T, H are collinear. By similarity, $\angle ESF = \angle GSH$. By the inscribed angle theorem, $\angle ESF = \angle ETF$ and $\angle GSH = \angle GTH$, so $\angle ETF = \angle GTH$. By the vertical angles theorem, E, T, G are collinear. Thus, by both collinearities, T is the bi-medial of $\overline{EFGH}$. S and T are distinct; so, by triangle frustum theorem III, $\overleftrightarrow{EF} \nparallel \overleftrightarrow{GH}$. ■

We will call the difference of the base angles of a triangle its **skew angle**.

Construction 4.18
Construct a triangle given its circumradius, the sum of its legs and its skew angle.

Solution
If the skew angle is zero, then the triangle is isosceles. The legs are half the sum of sides; draw a circle of the given radius and inscribe the triangle in it. The problem is solved.

If the skew angle is not zero, estimate where to cut the sum of sides with $\overline{GE} < \overline{FG}$ and inscribe $\overline{EFG}$ in a circle of the given radius so $\overline{FG} + \overline{GE}$ is the sum of legs. Find J on the circle so $\overleftrightarrow{EF} \parallel \overleftrightarrow{GJ}$. By the parallels and circle theorem, $\overline{EG} = \overline{FJ}$, so $\overline{EFJG}$ is either a rectangle or an isosceles triangle frustum. By the rectangle or isosceles triangle frustum theorem, $\overline{EJ} = \overline{FG}$. Thus, the sum of the legs of $\overline{JGE}$ equals the given sum of $\overline{EFG}$ legs.

$\angle GEJ = \angle GEF - \angle JEF$	Addition
$= \angle GEF - \angle EJG$	Transversal theorem
$= \angle GEF - \angle EFG$	Inscribed angle theorem

$\angle GEJ$ is the given skew angle. By the inscribed angle theorem, inscribing $\angle GEJ$ in the circle of given radius fully defines $\overline{GJ}$, so we have the base, apex angle and sum of legs of $\overline{JGE}$. Construct it by C. 4.2. By C. 3.3, find F on the circle to get $\overline{EFG}$. ■

The hypothesis is that C. 4.2 is relevant. If given a triangle construction problem on an open-book exam, it is always a good idea to review those that are in the book. You might find it! If all that is being asked of you is to open your textbook to the correct page and copy the solution onto your exam paper, it would be lame to miss the question. In this case, C. 4.2 is the only solved triangle construction that requires the sum of legs. It also requires the base and the apex angle. Since you are given the circumradius, the next hypothesis is that the inscribed angle theorem is relevant. Easy! But the reality is that most students just sit and stare with no idea what to do. The discussion is that there are two solutions, one with $\overline{GE} < \overline{FG}$ and one with $\overline{FG} < \overline{GE}$.

Construction 4.19
Construct a triangle given its circumradius, the sum of its legs and the sum of its base angles.

Being given the sum of the base angles makes it easy. Being given the difference is interesting because this angle can be constructed elsewhere, and it has practical applications in engineering.

Skew is a term that is used in many disciplines with various meanings but – generally speaking – it refers to things that are not symmetrical around a vertical line. In geometry, the only triangle with zero skew is the isosceles triangle, which is symmetrical around its center line. Bridge trusses are usually isosceles triangles except when one riverbank is higher than the other; bridges that tilt are called skew bridges. Having more weight on one end than the other is complicated.

The triangle area theorem could be restated to say that, if the base and the height remain unchanged, skewing a triangle does not affect its area. This result is preliminary to the design of skew bridges, but they are considered advanced problems in structural engineering and are beyond the scope of this book. We will construct triangles given the skew angle and other data. Recall the notation $G^* := \overrightarrow{GI} \cap \overline{EF}$, the intersection of the apex angle bisector with the base.

Skew Angle Theorem

The angle between the altitude from the apex and the circumdiameter through the apex is equal to the skew angle. It is bisected by the apex angle bisector and the difference of the angles that this bisector makes with the base is also the skew angle.

Proof

Given $\overline{EFG}$ acute with G' the foot of the apex altitude and G'' diametrically opposed to G, assume that G' is on $\overline{EM_{EF}}$; if it is not, switch E and F. By the inscribed angle theorem, $\angle E = \angle GG''F$. By Thales' diameter theorem and complementarity, $\angle EGG' = \angle FGG''$.

$\angle G'GG'' = \angle G - 2(\rho - \angle E)$	Addition of angles at the apex
$= \angle G + 2\angle E - \sigma$	Expand
$= \angle G + 2\angle E - \angle E - \angle F - \angle G$	Angle sum theorem
$= \angle E - \angle F$	Simplify

By subtracting $\angle EGG'$ and $\angle FGG''$ from $\angle G$, its bisector also bisects $\angle G'GG''$.

$\angle FG^*G = \sigma - \angle F - \angle FGG^*$	Angle sum theorem
$\angle EG^*G = \sigma - \angle E - \angle EGG^*$	Angle sum theorem
$\angle FG^*G - \angle EG^*G = \angle E - \angle F + \angle EGG^* - \angle FGG^*$	Subtraction
$= \angle E - \angle F$	Definition of angle bisector

Proof of this for $\overline{EFG}$ obtuse is analogous. ■

Structural engineers call $\angle G'GG''$ the skew angle while we call $\angle E - \angle F$ this; it is the same thing. In these constructions, we will assume that $\angle F < \angle E$, but can always re-label E and F if needed.

Construction 4.20

Construct a triangle given the apex angle bisector, the altitude, and the median to the base.

Solution

By HL, construct $\overline{G^*GG'}$ and $\overline{M_{EF}GG'}$. By C. 1.5, replicate $\angle G^*GG'$ on the other side of $\overrightarrow{GG^*}$. By the skew angle theorem, this ray goes through the circumcenter. By the circumcenter theorem, the mediator of $\overline{EF}$ also goes through the circumcenter, which locates it. Draw a circle around it through G. $\overleftrightarrow{G'M_{EF}}$ intersects the circle at E and F. ■

The next construction is easy, so I will just outline the three solutions.

Construction 4.21

Construct a triangle given its circumradius, its skew angle and (1) the median to the base, (2) the apex altitude, or (3) the apex angle bisector.

Solution

1. Draw the circumcircle around O with radius R. Choose G and draw a ray the skew angle off $\overrightarrow{GO}$. Draw a parallel to it through O. Find M_{EF} on it the given length from G.
2. By the skew angle theorem and ASA, construct $\overline{G'GG''}$ with G'' the intersection of $\overrightarrow{GO}$ and $\overleftrightarrow{EF}$. From G, lay off R to O; draw the circumcircle, ω. $\overleftrightarrow{G'G''}$ cuts ω at E and F.
3. The same as (2) but using AAS and half the skew angle to construct $\overline{G'GG^*}$. ■

Construction 4.22

Construct a triangle given the apex angle bisector, the apex altitude and the base.

Solution

By HL, construct $\overline{G^*GG'}$. Guess at E and F on $\overleftrightarrow{G'G^*}$ so $\overline{GE} < \overline{FG}$. Let $E_{GG'}$ be the reflection of E around $\overleftrightarrow{GG'}$. Thus, $\overline{EGE_{GG'}}$ is isosceles and, by the isosceles triangle theorem, $\angle G'E_{GG'}G = \angle E$. $\angle E_{GG'}GF = \angle G'E_{GG'}G - \angle F = \angle E - \angle F$ by the exterior angle theorem. By the skew angle theorem, $\angle E_{GG'}GF = 2\angle G^*GG'$, which is known from the construction of $\overline{G^*GG'}$. Raise a perpendicular to $\overleftrightarrow{EF}$ from E towards G and extend $\overrightarrow{E_{GG'}G}$; call their intersection J. By mid-segment theorem #2 and HL, $\overline{JE_{GG'}E}$ can be constructed because G' is the midpoint of $\overline{E_{GG'}E}$ and $\overleftrightarrow{G'G} \parallel \overleftrightarrow{EJ}$, so $\overline{EJ} = 2\overline{GG'}$ and $\overline{GG'}$ is given.

We are given $\overline{EF}$; so, by SAS, construct the right triangle $\overline{EFJ}$. We must locate G and for this we have two loci: It is on the mediator of $\overline{EJ}$ and $\overline{FJ}$ subtends $\sigma - \angle E_{GG'}GF$, which we found to be $\sigma - 2\angle G^*GG'$. Construct this locus by P. 4.5. ■

The discussion is that there are two solutions, one with $\overline{GE} < \overline{FG}$ and one with $\overline{FG} < \overline{GE}$.

Mediator and Angle Bisector Theorem
The mediator of a chord and the bisector of an angle subtended by it meet on the circumcircle.

Proof
Given $\overline{EFG}$ with circumcenter O, let L be the intersection of a diameter through M_{EF} with the circumcircle, ω, on the other side of $\overleftrightarrow{EF}$ from G. By the diameter and chord theorem, $\angle EM_{EF}L = \angle FM_{EF}L = \rho$, so $\overline{EM_{EF}L} \cong \overline{FM_{EF}L}$ by SAS. Thus, $\overline{EL} = \overline{FL}$. By the subtend-at-center theorem, $\angle EOL = \angle FOL$, and, by the inscribed angle theorem, $\angle EGL = \angle FGL$. So, L is where the bisector of $\angle EGF$ intersects ω; that is, $L \equiv L_G$. Thus, L_G is both where the bisector of $\angle G$ and the mediator of $\overline{EF}$ intersect ω. ■

Angle Bisectors and Circumdiameter Theorem
The interior and exterior bisectors of a triangle's apex angle cut its circumcircle at the ends of a diameter that mediates its base.

Proof
Given $\overline{EFG}$, by the circumcenter theorem, the mediator of $\overline{EF}$ is a diameter and, by the mediator and angle bisector theorem, the interior bisector of $\angle G$ intersects it at endpoint L. By the interior and exterior angles theorem, the interior and exterior bisectors of $\angle G$ are perpendicular; so, by Thales' diameter theorem, the exterior bisector of $\angle G$ cuts the circumcircle at a point diametrically opposed to L. ■

Half the Skew Angle Theorem
The exterior bisector of a triangle's apex angle makes an angle with the extension of the base that is half the skew angle.

Proof
By the interior and exterior angles theorem, the exterior bisector of the apex angle is perpendicular to its interior bisector. By the pairwise perpendiculars theorem, the exterior bisector of a triangle's apex angle makes an angle with the extension of the base that is equal to the angle between the altitude from the apex and the apex angle bisector. By the skew angle theorem, this is half the skew angle. ■

Tangent and Exterior Bisector Theorem
Given $\overline{EFG}$, if the exterior bisector of $\angle G$ cuts $\overleftrightarrow{EF}$ at S and the tangent to the circumcircle at G cuts $\overleftrightarrow{EF}$ at T, then T is the center of the circle through S, G and G^*.

Construct a triangle with right vertex G that is rotated around G by half the skew angle from $\overrightarrow{GO}$.

Do you know how the *Tang–Soo–Do* seon-saeng [teacher] wants you to keep your feet shoulder width apart? Bruce Lee was the founder of *Jeet–Kune–Do*, but he said the same thing:

> *Many traditional classical stances assumed by martial artists are quite a sight. They range from exotic ballet-like stances to postures of squatting down in wide stances and grimacing as though laying an egg.* – Bruce Lee

Green belts of *Geometry–Do* have it easy compared to green belts of *Tang–Soo–Do*; not only is nobody trying to knock us unconscious, but, as the following theorem proves, for points on the circumcircle of a rectangle, the distance between the feet of the perpendiculars dropped on the diagonals is constant. We will call this distance between feet a rectangle's **shoulder width**.

Shoulder Width Stance Theorem
For any point on the circumcircle of a rectangle, the distance between the feet of the perpendiculars dropped on the diagonals is the altitude of the rectangle's definitional triangle.

Proof
By the rectangle theorem, perpendiculars dropped on diagonals from the vertices are equal; so, for $\overline{EFGH}$, medial point T, we need only consider points P on the arc from E to F. Let P', P'', E' be the feet of perpendiculars dropped onto $\overline{FH}, \overline{EG}, \overline{FH}$, respectively.

Case One: *$\overline{EFGH}$ is a square.*
By the Lambert theorem, $\overline{PP'TP''}$ is a rectangle. By the rectangle theorem, its diagonals, $\overline{TP} = \overline{P'P''}$, are equal. $\overline{TP} = \overline{TE}$ are both radii. By transitivity, $\overline{TE} = \overline{P'P''}$. •

Case Two: *$\overline{EFGH}$ is not a square and $\overline{PP'TP''}$ is a quadrilateral, with vertices in this order.*
$\overline{PP'TP''}$ is cyclic by the cyclic quadrilateral theorem converse. Draw a ray $\overrightarrow{TS}$ such that $\angle ETE' = \angle PTS = \alpha$ and S is the foot of the perpendicular dropped from P onto $\overrightarrow{TS}$. By Thales' diameter theorem, S is on the $\overline{PP'TP''}$ circle. By the inscribed angle theorem, $\angle P'P''P = \angle P'TP = \beta$ and $\angle P''SP = \angle P''P'P$. By AAS, $\overline{TE'E} \cong \overline{TSP}$, so $\overline{E'E} = \overline{SP}$.

$\angle SPP'' = \sigma - \angle STP''$	Cyclic quadrilateral theorem
$= \sigma - \angle PTP'' - \alpha$	Addition and substitute $\alpha = \angle PTS$
$= \sigma - (\sigma - \alpha - \beta) - \alpha$	α, β and $\angle PTP''$ are on a side of $\overleftrightarrow{FH}$
$= \angle P'P''P$	Simplify and substitute $\beta = \angle P'P''P$

By AAS, $\overline{SPP''} \cong \overline{P'P''P}$, so $\overline{SP} = \overline{P'P''}$. By transitivity, $\overline{E'E} = \overline{P'P''}$. •

If $\overline{FG} < \overline{EF}$ and P is near E or F, $\overline{PP'TP''}$ may not be a quadrilateral. This is left as an exercise.

Inscribed Angle Theorem Converse
If two equal angles with vertices on the same side of a segment are subtended by it, their vertices and the endpoints of the segment are corners of a cyclic quadrilateral.

Problem 4.11 *Given $\overline{EFGH}$ cyclic, its center O, and its bi-medial T, assume $P := \overrightarrow{EF} \cap \overrightarrow{HG}$ exists. Prove that the bisectors of $\angle P = \angle EPH$ and $\angle T = \angle ETF$ are perpendicular.*

Solution
If T is on the bisector of $\angle P$, the solution is easy. Assume that it is on the same side as G; if it is not, switch the F and G labels, and switch the E and H labels. Let M be the intersection of the bisectors of $\angle P$ and $\angle T$. Let J and K be the intersections of the bisector of $\angle P$ with $\overline{GE}$ and $\overline{FH}$, respectively, so $\angle J = \angle TJK$ and $\angle K = \angle TKJ$.

$\angle P = \frac{1}{2}(\angle EOH - \angle FOG)$	Intersecting secants angle theorem
$\quad = \angle EFH - \angle FHG$	Inscribed angle theorem
$\angle K = \angle EFH - \frac{1}{2}\angle P$	Exterior angle theorem and rearrange
$\quad = \angle EFH - \frac{1}{2}(\angle EFH - \angle FHG)$	Substitution
$\quad = \frac{1}{2}(\angle EFH + \angle FHG)$	Simplify
$\angle T = \sigma - \angle EFH - \angle FEG$	Angle sum theorem and extension
$\quad = \sigma - \angle EFH - \angle FHG$	Inscribed angle theorem
$\angle J = \sigma - \angle T - \angle K$	Angle sum theorem and rearrange
$\quad = \sigma - (\sigma - \angle EFH - \angle FHG) - \frac{1}{2}(\angle EFH + \angle FHG)$	Substitution
$\quad = \frac{1}{2}(\angle EFH + \angle FHG)$	Simplify

Thus, $\angle J = \angle K$. By the isosceles triangle theorem converse, $\overline{JTK}$ is isosceles and, by the center line theorem, $\angle M$ is right. ■

Cyclic Quadrilateral Mediators Theorem
A quadrilateral is cyclic if and only if the mediators of any three of its sides are concurrent.

Proof
Assume that $\overline{EFGH}$ is cyclic, so $\overline{EO} = \overline{FO} = \overline{GO} = \overline{HO}$. Thus, by the mediator theorem, the center, O, lies on all four of the mediators.

Given quadrilateral $\overline{EFGH}$, assume that the mediators of $\overline{EF}, \overline{FG}$ and $\overline{GH}$ are concurrent at O. By SAS, $\overline{M_{EF}EO} \cong \overline{M_{EF}FO}$ and $\overline{M_{FG}FO} \cong \overline{M_{FG}GO}$ and $\overline{M_{GH}GO} \cong \overline{M_{GH}HO}$, which holds the equalities, $\overline{EO} = \overline{FO} = \overline{GO} = \overline{HO}$. Thus, $\overline{EFGH}$ is cyclic. ■

Like the Lambert theorem, this seems inconsequential until you realize that you can rarely prove all four mediators concurrent, but you often have three of them concurrent, and that is sufficient.

In general, a sum of segments can only be compared to a longer segment if you can get their lengths laid off on the longer segment. In the following problem, $\overline{EF}$ and $\overline{HG}$ each share an endpoint with $\overline{FG}$, so $\overline{FG} = \overline{EF} + \overline{HG}$ only if there is J on $\overline{FG}$ such that $\overline{EFJ}$ and $\overline{HGJ}$ are isosceles.

Problem 4.12 *$\overline{EFGH}$ is cyclic. There is also a circle that has its center, O_1, on $\overline{FG}$ and touches $\overrightarrow{EF}$, $\overrightarrow{HG}$ and $\overline{EH}$. Prove that $\overline{FG} = \overline{EF} + \overline{HG}$.*

Solution

By the two tangents theorem, O_1 is on the bisectors of $\angle E$ and $\angle H$. Let J be on $\overline{FG}$ such that $\overline{JG} = \overline{HG}$. We will do case one, J is inside $\overline{O_1G}$, and leave the other cases as exercises.

By the isosceles angle theorem, $\angle GJH = \rho - \frac{1}{2}\angle G$. By the cyclic quadrilateral theorem, $\angle E + \angle G = \sigma$; thus, $\frac{1}{2}\angle G = \rho - \frac{1}{2}\angle E$. By substitution, $\angle GJH = \frac{1}{2}\angle E = \angle O_1EH$. But $\angle GJH + \angle O_1JH = \sigma$, so $\angle O_1EH + \angle O_1JH = \sigma$ and, by the cyclic quadrilateral theorem converse, $\overline{EO_1JH}$ is cyclic. By the inscribed angle theorem, $\angle O_1HE = \angle O_1JE$, which is $\angle O_1HE = \angle FJE$ by extension. But $\angle F + \angle H = \sigma$ by the cyclic quadrilateral theorem, which implies $\frac{1}{2}\angle H = \rho - \frac{1}{2}\angle F$. By substitution, $\angle FJE = \rho - \frac{1}{2}\angle F$ and, by the angle sum theorem, $\angle JEF = \rho - \frac{1}{2}\angle F$. By the isosceles triangle theorem converse, $\overline{EJF}$ is isosceles with $\overline{FJ} = \overline{EF}$. Thus, $\overline{FG} = \overline{FJ} + \overline{JG} = \overline{EF} + \overline{HG}$. •

Orthocenter and Circumcircle Theorem

H is the orthocenter of $\overline{EFG}$ if and only if its reflections around the sides are on the circumcircle.

Proof

Let H_E, H_F, H_G be the pedal points of H in $\overline{EFG}$ and H_{FG} be the reflection of H around $\overleftrightarrow{FG}$. By SAS, $\overline{HH_EG} \cong \overline{H_{FG}H_EG}$, so $\angle H_EHG = \angle H_EH_{FG}G$. By the right cyclic theorem and supplementarity, $\angle H_EHG = \angle H_GFH_E$. Thus, $\angle EH_{FG}G = \angle EFG$. By the inscribed angle theorem converse, $\overline{EFH_{FG}G}$ is cyclic. Analogously for $\overline{EFGH_{GE}}$ and $\overline{EH_{EF}FG}$. •

Proof of the converse is left as an exercise; H_{EF}, H_{FG}, H_{GE} will be called **orthic reflections**.

Orthocenter and Circumcenter Theorem

Given $\overline{EFG}$ with $\overline{EG} \neq \overline{FG}$, orthocenter H, and G'' diametrically opposed to G in the circumcircle, then $\overleftrightarrow{EF} \parallel \overleftrightarrow{H_{EF}G''}$.

The orthocenter is $H := \overleftrightarrow{EE'} \cap \overleftrightarrow{FF'} \cap \overleftrightarrow{GG'}$ in $\overline{EFG}$. For $\overline{EFG}$ not right, if $\angle G < \rho$, then $\overline{EG'HF'}$, $\overline{FE'HG'}$ and $\overline{EFE'F'}$ are cyclic; if $\angle G > \rho$, then $\overline{EG'GE'}$, $\overline{FF'GG'}$ and $\overline{EFF'E'}$ are cyclic. The reasoning is analogous, but the order of the vertices is different depending on whether H is inside or outside $\overline{EFG}$. When working with orthocenters, failure to test your proof on acute, right, and obtuse triangles is a pitfall. Incenter and medial point theorems are less treacherous!

Problem 4.13 *Given $\overline{EFG}$ and its orthocenter H, prove the following:*

1. *Any one of E, F, G, H is the orthocenter of the triangle whose vertices are the other three.*
2. *The four triangles whose vertices are any three of E, F, G, H all have equal circumcircles.*
3. *If four equal circles intersect in four points, E, F, G, H, then H is the orthocenter of $\overline{EFG}$.*
4. *$\overline{EFG} \cong \overline{O_1O_2O_3}$ with O_1, O_2, O_3 the circumcenters of $\overline{FHG}, \overline{GHE}, \overline{EHF}$, respectively. Also, if you swap H with E, F or G and the circumcenter of $\overline{EFG}$ with O_1, O_2 or O_3, respectively.*

Problem 4.14 *Given $\overline{EFGH}$ cyclic and I_E, I_F, I_G, I_H the incenters of $\overline{EFH}, \overline{FGE}, \overline{GHF}, \overline{HEG}$, respectively, prove that $\overline{I_EI_FI_GI_H}$ is a rectangle.*

Prove that $\overline{EFI_FI_E}$ and $\overline{FGI_GI_F}$ are cyclic, find $\angle FI_FI_E + \angle FI_FI_G$ in terms of $\angle E$ and $\angle G$, which are supplementary, so $\angle FI_FI_E + \angle FI_FI_G = 2\sigma - \rho$ and $\angle I_EI_FI_G = \rho$ by **closing the horizon**.

Quadrilateral Angle Bisectors Theorem

The bisectors of the external angles of a quadrilateral form a cyclic quadrilateral.

Proof

Given quadrilateral $\overline{EFGH}$, let $P_{EF}, P_{FG}, P_{GH}, P_{HE}$ be the intersections of the external angle bisectors such that $\overline{EFP_{EF}}$, $\overline{FGP_{FG}}$, $\overline{GHP_{GH}}$, $\overline{HEP_{HE}}$ are triangles built on the sides of $\overline{EFGH}$ and exterior to it. $P_{EF}, P_{FG}, P_{GH}, P_{HE}$ are outside $\overline{EFGH}$ because, in *Geometry–Do*, all quadrilaterals are convex. In this theorem, P_{EF} is *not* the reflection of P around $\overleftrightarrow{EF}$, so our notation defies our convention, but subscripts seemed the clearest way to say this.

$\angle EP_{EF}F = \sigma - \frac{1}{2}(\sigma - \angle E) - \frac{1}{2}(\sigma - \angle F)$	Angle sum theorem
$\angle EP_{GH}F = \sigma - \frac{1}{2}(\sigma - \angle G) - \frac{1}{2}(\sigma - \angle H)$	Angle sum theorem
$\angle EP_{EF}F + \angle EP_{GH}F = 2\sigma - \frac{1}{2}\big(4\sigma - (\angle E + \angle F + \angle G + \angle H)\big)$	Addition
$\angle EP_{EF}F + \angle EP_{GH}F = 2\sigma - \frac{1}{2}(4\sigma - 2\sigma)$	Quadrilateral angle sum theorem
$\angle EP_{EF}F + \angle EP_{GH}F = \sigma$	Subtraction

$\overline{P_{EF}P_{FG}P_{GH}P_{HE}}$ is cyclic by the cyclic quadrilateral theorem converse. ■

It is sometimes true that the bisectors of the internal angles of a quadrilateral form a cyclic quadrilateral, but stating this as a theorem would require caveats, like for squares and kites. I will just leave the student with a reminder to consider this if given a problem that mentions all four internal angle bisectors. The proof is analogous, but with use of the vertical angles theorem.

Finally, let us conclude this chapter with something fun; how to be a *malvoisine*, or a bad neighbor. Imagine that this is the Middle Ages and you are laying siege to a walled city, your only projectile weapon being the trebuchet. The city walls are square, with the citadel in the center. The trebuchet is a large device that cannot traverse once emplaced. The beaten zone is the area where most of the stones fall, and this can be moved forwards or backwards by launching different sized stones. A sextant can be used to precisely measure the angle between two objects, but there is no means of measuring distances while under fire. You must attack at a slant because the defenders have trebuchets prepositioned to fire down the mediator of each wall.

Problem 4.15 *Using only a sextant, position a trebuchet so it fires directly at the citadel in the center of a square walled city; you cannot see over the wall and have no distance measurements.*

> ***Solution***
> Push the trebuchet forward until you find a spot where the wall subtends a right angle. By the cyclic quadrilateral theorem converse, you, the corners of the wall and the citadel are concyclic. By the mediator and angle bisector theorem, the bisector of your right angle with the corners is a straight shot at the citadel. Thus, you want the angles between your trebuchet's aiming point and each corner of the wall to be half of a right angle. ■

If you know that you will always be attacking square forts, a sextant is unnecessary. Mount boards on the front of the trebuchet aiming $\frac{1}{2}\rho$ to each side. Mount them so the gunner can sit behind the ballast and straddling the projectile chute to look down each board like a gunsight.

Problem 4.16
You are sneaking up on the Pentagon with a trebuchet in what must be the most ill-conceived act of terrorism ever. *How do you use a sextant to aim for the facility's center?*

By construction 3.19, you can construct a regular pentagon for use in solving this problem.

Problem 4.17
The enemy has three antiaircraft guns in an equilateral triangle with a munitions dump at the center. Afraid to attack from the air, you are sneaking up on it with a self-propelled mortar. But you are afraid to reveal your position with a laser rangefinder, so you plan to aim over the munitions dump and then walk your shells back until you hear a secondary explosion. How?

Green Belt Exit Exam

1. *$\overline{EFGH}$ is cyclic and orthodiagonal; G'' is diametrically opposite G. Prove that $\overline{HG''} = \overline{EF}$.*

2. *Given $\overline{EFG}$ with orthocenter H, prove that the circumradii of $\overline{EFH}, \overline{FGH}$ and $\overline{GEH}$ equal that of $\overline{EFG}$.*

3. *Given $\overline{EFG}$ and J, K, L, construct $\overline{E''F''G''} \cong \overline{EFG}$ such that J is on $\overline{E''F''}$, K is on $\overline{F''G''}$ and L is on $\overline{G''E''}$. J on $\overline{E''F''}$ can also be written using set theory notation, $J \in \overline{E''F''}$.*

4. *Given $\overline{EFGH}$ cyclic but not a kite with H_{EFG} and H_{EGH} the orthocenters of $\overline{EFG}$ and $\overline{EGH}$, respectively, prove that $\overline{H_{EGH}H_{EFG}FH}$ is a parallelogram.*

5. *Given $\overline{EFG}$ with orthocenter H, find P on the circumcircle such that $\angle EFP = \rho$. Prove that $\overline{GH} = \overline{FP}$.*

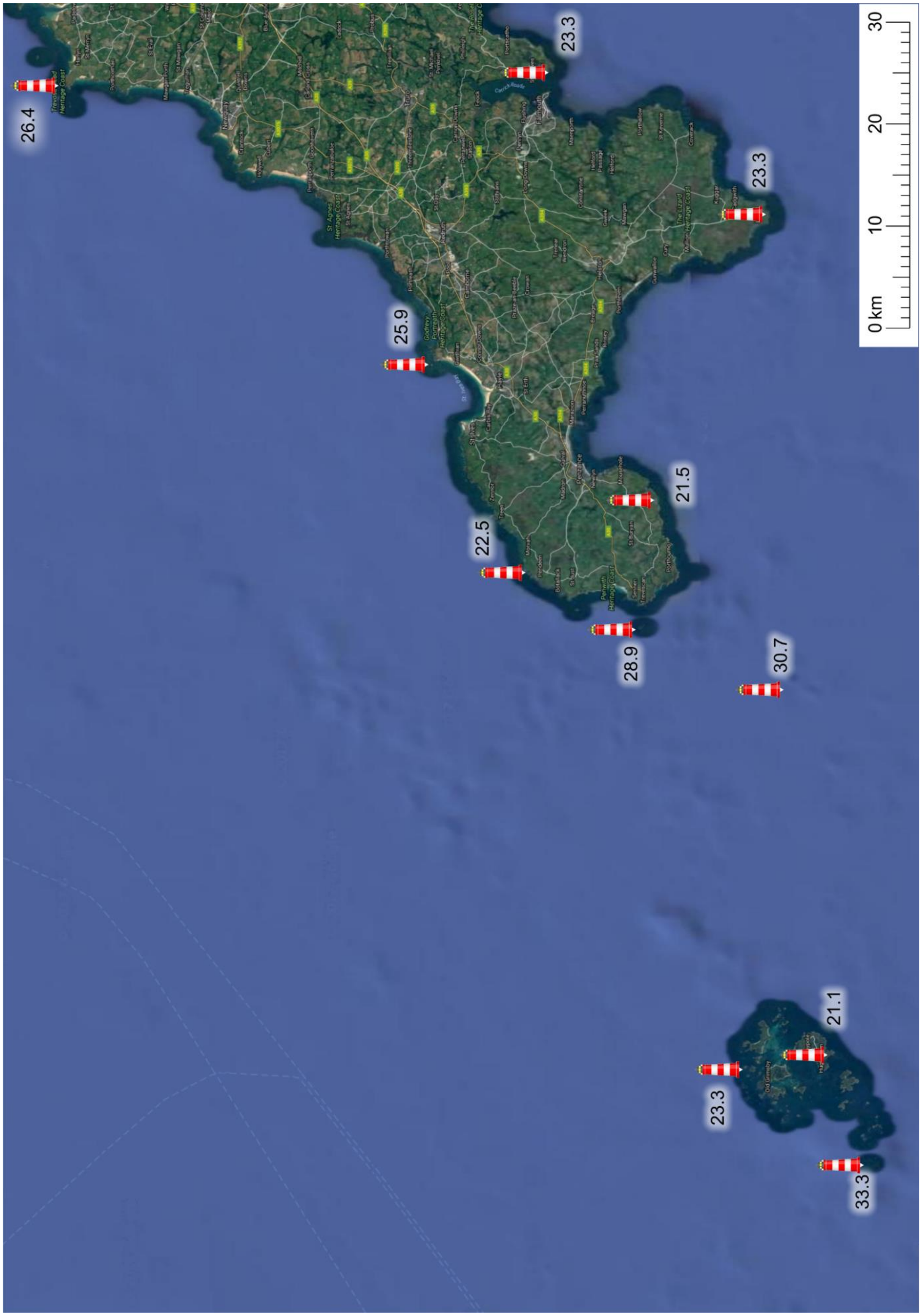
6. *Prove that the centers of equilateral triangles internal to triangle sides form an equilateral triangle.*

7. *Construct a triangle $\overline{EFG}$ given its base $\overline{EF}$ and its base vertices' altitude feet, E' and F'.*

8. ***Johnson Theorem***
   If three equal circles are concurrent, then their other three intersections define a circle of the same radius.

9. ***Japanese Theorem***
   When Roger Johnson visited Japan, he found, drawn on a temple wall, two figures of the same cyclic polygon, but partitioned into triangles differently. In both cases, the inradii of the triangles had the same sum! Prove this for a cyclic quadrilateral cut by either diagonal.

10. *You are the captain of a ship out of Dublin and bound for London. As you approach the Cornwall peninsula counterclockwise (See the map on the next page; the range of each lighthouse is printed beside it.), you become disoriented in heavy fog. The lookout reports that he can see three lighthouses, one dead ahead. With a sextant, he measures the angle between the port and bow lighthouses as $34°$, and the angle between the starboard and bow lighthouses as $98°$. He does not know which lighthouses he is seeing; indeed, there may be others in the vicinity that he does not see. Plot a course around the peninsula.*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Practice Problems

4.18 Construct a triangle given the following information:

1. *The apex angle, apex altitude, and apex angle bisector*
2. *The base, the difference of the sides, and the altitude to a side*
3. *The base, the apex angle, and the sum of the altitudes to the sides*
4. *A base angle, the base, and the sum of the legs*
5. *A base angle, the apex altitude, and the perimeter*
6. *The apex angle, the apex altitude, and the perimeter*
7. *A base angle, the altitude to one leg, and the sum of the legs*
8. *The apex angle and the medians to the legs*

4.19 Construct a parallelogram given the following information:

1. *An angle and the lengths of both altitudes*
2. *The lengths of both altitudes and a diagonal*
3. *The lengths of an altitude and both diagonals*
4. *The lengths of one of its constituent triangles*
5. *A side and a diagonal; also, the angle between them*

4.20 Construct a quadrilateral, $\overline{EFGH}$, given the following information:

1. $\angle E, \angle F, \angle G$ *and the lengths of* $\overline{EF}$ *and* $\overline{EH}$.
2. $\angle F, \angle G$ *and the lengths of* $\overline{EF}, \overline{FG}$ *and* $\overline{GH}$.
3. $\angle E, \angle F, \angle G$ *and the lengths of* $\overline{EH}$ *and* $\overline{GH}$.

4.21 Find the point that minimizes the sum of the distances to the vertices of a quadrilateral.

4.22 Prove that parallelism is an equivalence relation. (Euclid, Book I, Prop. 30)

4.23 Given $\overline{EFGH}$, E, J, K, F collinear, $\angle JHE = \angle FGK$ and $\angle KHJ = \angle KGJ$; prove that $\angle JGE = \angle FHK$.

4.24 Prove that the sum of the legs of a right triangle equals the sum of the indiameter and the circumdiameter.

4.25 Prove that, if a circle cuts equal segments from the sides of a quadrilateral, then the quadrilateral is tangential.

4.26 Given $\overline{EFG}$ with circumcircle ω, let $G'' := \overrightarrow{HM_{EF}} \cap \omega$. Prove M_{EF} is the midpoint of $\overline{HG''}$.

4.27 *Prove that the altitude and the diameter of the circumcircle from the apex cut a chord from the circumcircle that is parallel to the base.*

4.28 *Given $\overline{EFG}$, draw a circle, ω, around H through G. Let $P := \overrightarrow{GE} \cap \omega$ and $Q := \overrightarrow{GF} \cap \omega$. Prove that E, P, H, Q, F are concyclic.*

4.29 *Given a circle, ω, and a point, H, inside it, construct a triangle $\overline{EFG}$ such that ω is its circumcircle and H is its orthocenter. Is $\overline{EFG}$ fully defined?*

4.30 *Given $\overline{EFG}$ with $\angle G = \varphi$, prove that $\overline{HI} = \overline{OI}$.*

4.31 *Given $\overline{EFG}$ acute, raise a perpendicular to $\overleftrightarrow{OM_{GH}}$ through M_{GH}. Let P and Q be its intersections with $\overline{GE}$ and $\overline{FG}$, respectively. Prove that $M_{PQ} \equiv M_{GH}$.*

4.32 *Given $\overline{EFG}$ isosceles with base $\overline{EF}$, if G'' is the foot of the perpendicular dropped on the line tangent to the circumcircle at F, prove that $\overline{GG'} = \overline{GG''}$.*

4.33 *Given overlapping circles ω_1 and ω_2 with common chord $\overline{EF}$ and a line tangent to ω_1 and ω_2 at T_1 and T_2, respectively, prove that $\angle T_1ET_2 + \angle T_1FT_2 = \sigma$.*

4.34 *Given overlapping circles ω_1 and ω_2 with an intersection at P, draw an arbitrary line through P that intersects ω_1 and ω_2 at T_1 and T_2, respectively. Prove that the tangents to ω_1 at T_1 and to ω_2 at T_2 make an angle that is constant for any line drawn through P.*

4.35 *Given $\overline{EFG}$ with $\angle F < \angle G$ and L''_E diametrically opposed to L_E, prove that $\angle L''_E L_E E = \frac{1}{2}(\angle G - \angle F)$.*

4.36 *Given $\overline{EFG}$ with P any point on $\overline{EF}$, draw a circle through E and P that is tangent to $\overleftrightarrow{EG}$ at E, and another circle through F and P that is tangent to $\overleftrightarrow{FG}$ at F. Prove that these two circles and the circumcircle of $\overline{EFG}$ are concurrent.*

4.37 *Given $\overline{EFGH}$ cyclic and $P := \overleftrightarrow{EF} \cap \overleftrightarrow{GH}$ and $Q := \overleftrightarrow{FG} \cap \overleftrightarrow{HE}$ and M the other intersection besides F of the circles through F, G, P and through E, F, Q, prove P, M, Q to be collinear.*

4.38 *Given $\overline{EFG}$, prove that the circumcircles of $\overline{M_{GE}EM_{EF}}$ and $\overline{M_{EF}FM_{FG}}$ and $\overline{M_{FG}GM_{GE}}$ are concurrent at the circumcenter of $\overline{EFG}$. This is a special case of the Miquel theorem.*

Rules and Tactics for a Trebuchet and Paintball Battle

At this point in a young mathematician's career, it is traditional for the mathletes to challenge the JROTC to a paintball battle. Each team is armed with paintball guns and two trebuchets that throw five-pound bags of flour or clusters of one-pound bags for anti-personnel use. Use 25# weightlifter's plates for ballast; for safety's sake, agree to an upper limit on the number of plates.

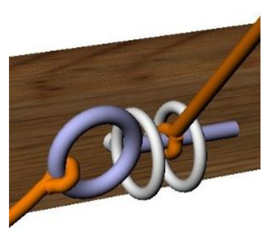
Facing each other and about twice as far apart as the maximum range of a trebuchet, construct two 16' square forts out of plywood with gates in their forward walls. Each fort has a 4'x4' sheet of plastic on the ground in the exact center. To win you must drop a 5# bag of flour on the enemy's plastic sheet, either by firing it from a treb or by overrunning the fort and hand-dropping a bag of flour on the plastic. The students may use chalk lines of any length, but only a single 16' tape measure. When complete, their forts are tested for squareness with a 25' tape measuring the diagonals, which should be 22' 7.5". If a team's fort is not perfectly square, then they are penalized by taking some of their ammunition away from them before the battle begins.

Show the JROTC sergeant the above paragraphs, but do not show him the following hints. He should already know tactics, but math teachers may not have this background, so they get hints.

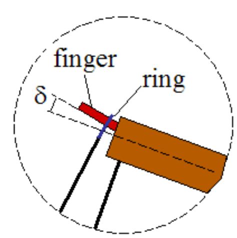
1. Your forward treb should have plywood armor on one side and tow ropes on the other. By pulling sideways and keeping the armor facing backwards, your rear treb can fire clusters of one-pound bags of flour directly at the forward treb to clear away enemy that approach it. (In Vietnam, two vehicles would fire flechettes at each other when overrun.)

2. Do not pull the forward treb directly towards the enemy fort because that is probably what they will do, and you want to flank them. Have the rear treb to one side of your fort so it is aimed at the forward treb as you pull it forwards at an angle to the enemy fort.

3. Use problem 4.15 for emplacing the trebuchet to aim directly towards the center of the enemy fort. The pivot should be about at the quartile point; the ballast must be on a cable to prevent rocking back and forth. The angle of elevation can be adjusted by changing the angle of the pin, δ, that releases the sling. Have a chute for the pouch to slide in.



Gunner's Trigger



Release Trigger

This is a good reference except that they have the ballast hinged on the beam and wheels for the rocking motion. www.real-world-physics-problems.com/trebuchet-physics.html

Machine Gun Emplacement as an Application of Geometry

Top traverse is an angle, the lateral limit of a gun's ability to traverse. A gun's **field of fire** are all the points inside this angle; this is equivalent to the more geometric term visible under the angle. If this angle is interior to a triangle, the **kill chord** is the opposite side. If bunkers are to be built to cover a bottle neck, defined by the kill chord, then they must be positioned on an arc whose center is on the mediator of the kill chord such that the angle subtended at the center by the kill chord is twice that of the gunners' top traverse as defined by the slits in their identical bunkers.

Recall problem 4.5: Find the locus of vertices for a given angle subtended by a given chord.

We assumed that the top traverses of each gun are fixed and equal, but the guns' positions are variable. This is realistic as bunker construction is standardized and, since the locus is a long arc, we can choose positions on it that have non-geometric assets, like rock outcroppings. But now we will assume that the positions are fixed, like the only two rock outcroppings in the vicinity, and we will define their **kill circle**[88] to instruct the gunners on how much they should traverse.

If two guns have the same top traverse and the angle bisectors of these angles intersect at a point equally distant from both guns, that point is the center of the incircle of the quadrilateral over which their fields of fire overlap. (Verify.) We will call this the kill circle of the two guns, for it is the circle of largest radius in which every interior point is within the field of fire of both guns.[89] But, if the positions of the guns are chosen due to non-geometric assets, like rock outcropping, then they will not be equally distant from the center of their kill circle. To make the quadrilateral over which their fields of fire overlap tangential, the far gunner must be instructed to traverse over an angle less than his top traverse, which concentrates their fire on a circle. By defining the gunners' kill circle, geometry clearly defines their mission; they are not just spraying.

The near gunner defines the kill circle. Draw a ray from his position through the kill circle center and then draw rays on either side to define his top traverse. Drop perpendiculars from the kill circle center to these rays to find the touching points and then draw in the kill circle.

The far gunner must reduce his traverse. Draw a ray from his position through the kill circle center, bisect it and then draw an arc around its midpoint through the kill circle center. Where it intersects the kill circle are the touching points of rays that define his new top traverse.

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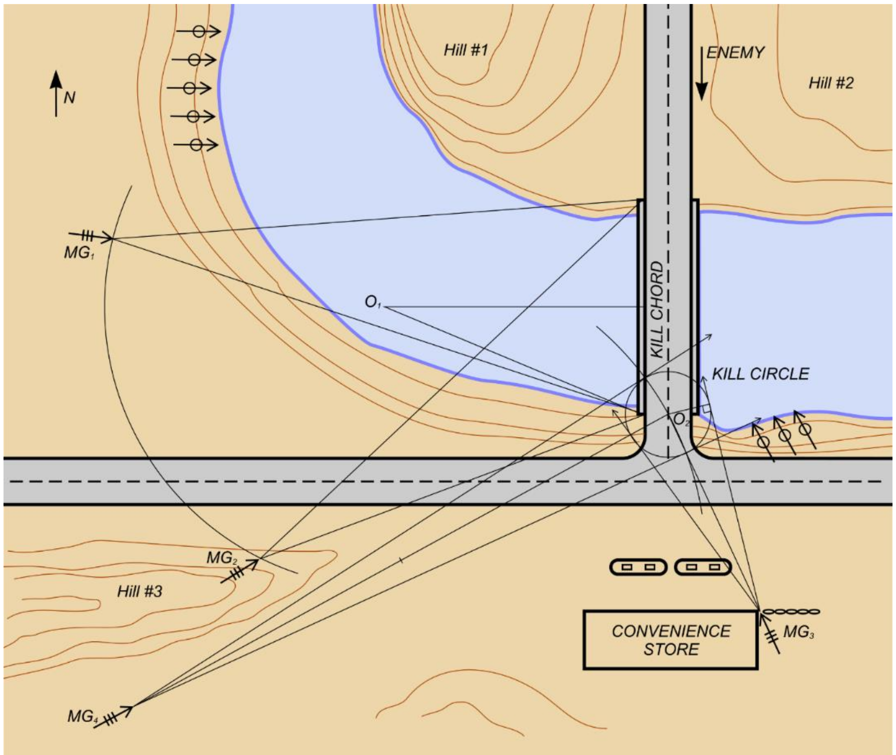
[88] I replace kill zone with two new terms: kill chord and kill circle. The kill chord is not inside the kill circle.

[89] The kill circle is not what beaten zone meant during WWI. This term is from the distant past when heavy machine guns were used like low-caliber artillery and fired at a high angle of elevation, so the bullets came down at a steep angle of descent. We are doing direct fire on armored vehicles at close range. Some third-world armies buy auto-cannons to shoot at first-world warplanes but, when they fight each other, they use them WWI-style against cities.



Scenario #1: The enemy comes in two BTR-80 armored personnel carriers, each armed with a 14.5 mm auto-cannon and coaxial 7.62 mm machine gun. They hope to secure the 100-meter bridge and seize the 10,000 liters of diesel at the store. You stand in their path with four M2 machine guns (top traverse: $22.5°$) and two dozen LAW rockets (effective range: 200 meters).

The kill chord of MG$_1$ and MG$_2$ is the bridge. Recalling that plumbers use a $5 : 12 : 13$ triangle to install $22.5°$ elbows, you construct one on half of the bridge, so the full bridge subtends a $22.5°$ angle at any point on the circle that contains this kill chord centered at the apex of the triangle, O_1. These gunners' greatest fear is a BTR-80 on hill #1, so grenadiers are positioned along the riverbank aiming for this hill. MG$_3$ can join this fight if the enemy delays rushing the bridge.

The kill circle of MG$_3$ and MG$_4$ is the bridge exit. MG$_3$ is at the side of the store and aimed for point O_2; its field of fire is centered on the ray to O_2. Drop a perpendicular from O_2 to one ray of this angle and then draw the kill circle. MG$_4$ is given to the best marksman and concealed behind hill #3 where it fires on the kill circle of MG$_3$. Bisect the segment from it to point O_2 and then draw an arc centered at this midpoint and passing through O_2 to find the points tangent to the kill circle. Draw rays to these touching points to determine how much MG$_4$ is to traverse. The grenadiers near the bridge are below the MG$_1$ and MG$_4$ fire grazing the bridge; also, they are below the -4° minimum elevation angle of the BTR-80 cannon. They are fighting Apache style![90]

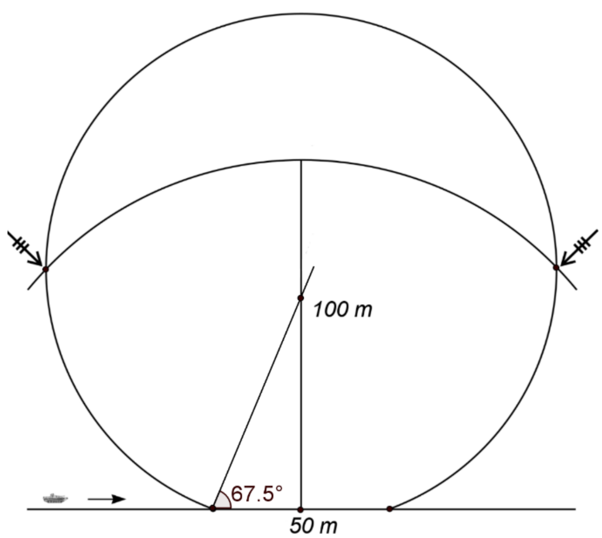
[90] Victorio's ambush at Cooke's Canyon on 29 May 1880 in New Mexico, USA is described by Watt (2012, p. 38).

The BTR-80 suspension was not designed by Lotus; speed bumps can really mess up their return fire! Concrete takes weeks to cure enough to stop 14.5 mm fire; speed bumps need only days. If the enemy comes in days, not weeks, do not make bunkers; install speed bumps on the bridge.

In this scenario, we took the top traverse to be $22.5°$ because, conveniently, this is an angle that I have already provided plumbers with instruction on how to construct. Protractors are forbidden in this textbook because real numbers have not been defined. But, since soldiers will typically read of the top traverse in their weapon's manual rather than be shown it, we will here allow the use of protractors for replicating and adding angles, just not multiplying them. If you subtract the top traverse from 90° and lay this angle off the endpoint of the kill chord with a protractor, the ray intersects the mediator at the center of the circle that contains the kill chord.

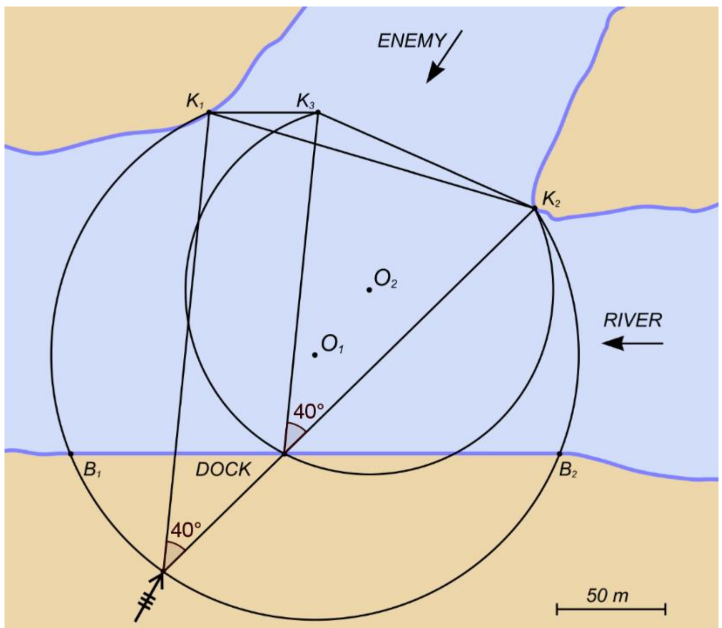
Scenario #2: We have four machine guns and wish to set an ambush from defilade, leading a BMP-2 that will drive across a 50-meter opening where vision is obscured on either side. Knowing that the 12.7 mm M2 machine gun is marginal against vehicles heavier than the BTR-80, the armorer has mounted them in coaxial pairs, so the bullets converge at a range of 100 meters.

Thus, there are two criteria for the emplacement of our two dual guns. We wish them to sweep the given kill chord within their given top traverse and we *also* wish their nearly coaxial fire to converge at the midpoint of the kill chord so the bullets striking the same spot at the given distance might penetrate armor plate that the 12.7 mm is not actually rated for. We are constructing a pair of triangles given the base, the apex angle, and the median to the base.



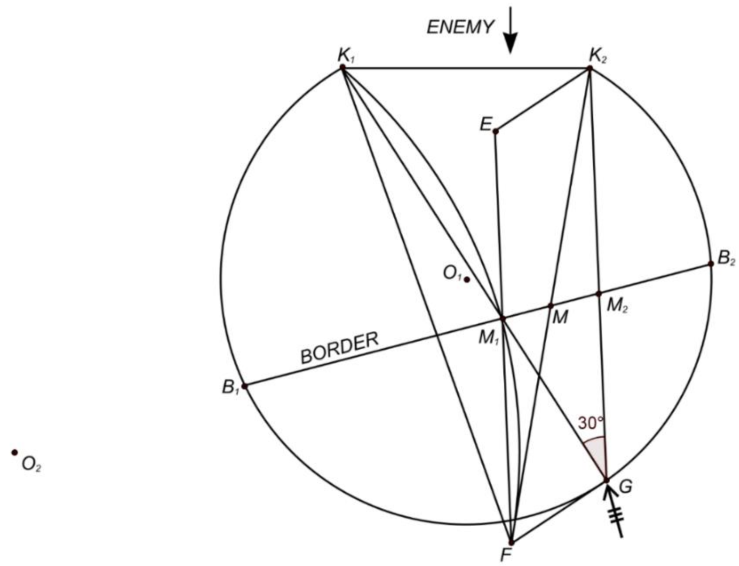
Scenario #3: Here we are fighting either in the past or in a modern country too poor to have any armored vehicles nor many machine guns. The border is defined by a river. At the confluence of it and another river that comes from your enemy's land, you, with only one machine gun, are tasked with defending against enemy marines who might come in small boats and attempt to cross the river under fire. On your bank of the river, you have much concertina wire. If you can kill half the enemy with your gun, your infantry can fight off the other half at the wire. However, during peacetime you trade with these people, and so there must be a 50-meter-wide break in the wire for a dock. To protect your gun from mortars, you will build a bunker with a 40° slit.

The soldiers wish the dock were narrower and the traders wish it were wider, so it must be *exactly* 50 meters to satisfy both interests. But it can be built anywhere on the chord defined by where your riverbank passes through the circle defined by the kill chord, which is defined by the width of the enemy river and by the 40° top traverse of the gun. The objective is to position it so the army's only machine gun can perform double duty: It must cover the kill chord across the enemy river, and it must *also* cover the dock that they hope to storm.



$\overline{K_1K_2}$ is the kill chord and O_1 is the center of the circle defined by the intersection of the kill chord mediator and a 40° angle laid against the kill chord. To avoid clutter, this triangle is not shown. B_1 and B_2 are the points on the riverbank between which your gun must be positioned on the O_1 arc to cover the kill chord. From K_1 draw a line parallel to $\overleftrightarrow{B_1B_2}$. Towards K_2, lay off a segment 50 meters long to K_3. This is one side of a parallelogram with the dock on the other side, but we still do not know any of the parallelogram's angles and so it cannot yet be drawn. By the transversal theorem, the two angles shown are equal and so $\overline{K_2K_3}$ also subtends a 40° angle. Find O_2 the same way O_1 was found and draw this circle. Where it intersects $\overleftrightarrow{B_1B_2}$ is the K_2 side of the dock. Circles typically intersect lines twice, as this one does, so the dock could have been positioned with its endpoint at the other intersection, but we chose to enfilade the enemy river.

Scenario #4: A straight fence defines the border between two countries, and, on the enemy side, geographic features define a bottleneck. Given a top traverse of $30°$, the kill chord drawn across the bottleneck defines an arc that your gun must be positioned on. There is a point on the fence, M, that one infantry platoon is tasked with defending to the left of and another platoon to the right of. To avoid the appearance of favoring one platoon over the other, you wish to position your gun so its field of fire covers equal segments of fence to the left and to the right of this point.



$\overline{K_1K_2}$ is the kill chord and O_1 is the center of the circle defined by the intersection of the kill chord mediator and a $30°$ angle laid against the kill chord. To avoid clutter, this triangle is not shown. B_1 and B_2 are the points on the border fence between which your gun must be positioned on the O_1 arc to cover the kill chord. Guess where the gun, G, is to be positioned and draw in its field of fire. Position M on the fence midway between the edges of the field of fire. We will draw a figure and then learn from it so we can draw it again starting with M at its given position and then finding G, rather than starting with G and then finding M, which we did for the first drawing.

If M is the bi-medial point of a parallelogram, $\overline{EFGK_2}$, then, by the parallelogram centroid theorem, it bisects any segment from one side to the other, including the segment of fence in the gun's field of fire, $\overline{M_1M_2}$, and the diagonal, $\overline{FK_2}$. Extend $\overrightarrow{K_2M}$ an equal distance to find F and then draw in the rest of $\overline{EFGK_2}$. By the transversal theorem, $\angle EM_1K_1$ is $30°$ and its supplement, $\angle FM_1K_1$ is $150°$. Thus, $\overline{FK_1}$ could be the kill chord for a second gun with the same top traverse of $30°$ positioned on an arc centered at O_2. Now let us redraw our figure with M in its given position. Extend $\overrightarrow{K_2M}$ an equal distance to find F; we cannot draw $\overline{EFGK_2}$ because we have only the diagonal. Connect $\overline{FK_1}$, find O_2 and then draw in the part of the O_2 circle on the other side of $\overleftrightarrow{FK_1}$. Where it intersects $\overline{B_1B_2}$ is M_1. Extend $\overrightarrow{K_1M_1}$ to find G on the O_1 circle. Connect $\overline{GK_2}$.

Surveying Techniques to Measure a Line Through an Obstacle

In *Surveying Techniques to Measure or Lay Off Lengths*, I describe how to extend a line by constructing a rectangle around a house, and to do so without a transit to measure angles. Transits are expensive and it takes a lot of time to level a tripod four times. In *Basic Terminology Used in Surveying*, I describe how a mortar gunner can locate a target on the other side of an office building that is providing concealment from the enemy's automatic cannon. This requires measuring angles with a sextant, not a transit, the use of which would be impractical in combat.

Problem 3.60

From your mortar, M, you extend a line 170 meters with an azimuth angle of $107°$ to F. Backsight and extend 170 meters to E. If G is an enemy gun to the north, $\angle EFG = 67°$ and $\angle FEG = 76°$, at what azimuth angle and range should the mortar gunner be instructed to fire his weapon? (It is best if maps are scaled so 1 cm is 10 m. Here, 5 mm equals 10 m fits on U.S. letter-size paper.)

In this section, I present a method for going around obstacles larger than buildings; but it is not for gunnery, because it requires already having collinear lines on both sides of the obstacle.

Suppose that two collinear lines approach the Fire Swamp, which you estimate is a little less than 400 m across. You consider using P. 3.60 to triangulate a point on the other line, but the Cliffs of Insanity prevent you from extending a line perpendicular to your path very far in either direction; there is just barely enough room to go around the swamp on either side. So, you call for a volunteer to act as your rod man and begin measuring your line directly through the swamp.

You are not 100 m into the Fire Swamp when your rod man is set upon by ROUS[91] and devoured. You may never get the sound of his anguished screams out of your head. It's horrible.

"Okay!" you announce, "I need another volunteer!"

"No! Never! We're not going to do it! You can't make us!" your mutinous work crew shouts as they close in on you, brandishing their pickaxes and shovels.

Afraid for your life, you fire your revolver in the air and shout, "Everybody calm down! I know exactly what to do. I have a plan. Nobody will have to enter the Fire Swamp."

You withdraw to your pickup, where your 14-year-old daughter is sitting in the cab listening to Hip Hop music while thumbing through her *Geometry–Do* textbook, reviewing theorems.

[91] Rodents of Unusual Size.

"I have absolutely no idea what to do," you confess, "No plan. Have you learned anything in that geometry class of yours that might help me out of this jam?"

"Oh sure," she replies, popping her bubble gum, "It's easy."

You order two men to set flags exactly 400 m from your transit on the left and on the right of the Fire Swamp. This requires walking within centimeters of the Cliffs of Insanity, so it would be impossible to get a transit out there to measure an angle – just the flags. From your transit at the end of the surveyed line, you measure the angle between the two 400-meter flags at $104°$.

You radio the survey party on the other side of the Fire Swamp and tell them to move their transit back and forth along their line until they measure the angle between the two flags at $128°$. Your daughter assures you that the distance between the two transits is now exactly 400 m. Why?

"Because the distances to the flags are equal, they are on a circle of radius 400 m, centered at your transit," she explains, "The **conjugate angle** of $104°$ is $360° - 104° = 256°$ and, by the inscribed angle theorem, at any point on the arc between the two flags, the angle between them is half this: $128°$. You want the intersection of the two loci: The other surveyed line and the arc."

Admittedly, this story is a bit unrealistic. The unrealistic part is not the ROUS – a fearsome creature once found in swamps throughout the world, and which persist today in New York City – it is unrealistic that you cannot set up a transit at either point to the sides of the Fire Swamp. If you can do that, then measure the distance to stations on either surveyed line, and the angle between them. By SAS, you have fully defined the triangle whose other side is the desired length.

The Law of Cosines is the trigonometric version of our SAS theorem: $g^2 = e^2 + f^2 - 2ef\cos\gamma$

This is how most surveyors would solve the problem of going around a swamp. There is nothing wrong with trigonometry; it is an interesting subject that can be quite challenging – thirty years after high school I still have nightmares about proving $\tan 2\theta = \frac{\sin\theta + \sin 3\theta}{\cos\theta + \cos 3\theta}$. However, this is a book about geometry, so I here present a problem with a geometric solution.

If you are plagued with trigonometers who believe that their science is useful in engineering while yours is just playing with axioms to no practical purpose, then sketch – not too accurately to prevent measuring it – $\overline{EFGH}$ with $|\overline{EF}| = |\overline{FG}| = 400$ m and $\angle F = 104°$ and $\angle H = 128°$. Then ask Mr. Smart-Aleck Trigonometer to find the length of $\overline{FH}$. That should shut him up!

On the subject of conjugate angles, this was once an SAT problem! Given E, F, G on an O-circle with O inside $\angle EGF$, prove that $\angle EOF = 2(\angle OEG + \angle OFG)$. What if O is not inside $\angle EGF$?

The Green Belt's Guide to Trigonometry

In *Elementary Quadrature Theory*, I noted that trigonometry has been eliminated and Geometry has been turned into a review of Algebra I with three teachers, of Algebra I, II and Geometry, being told to toss some trigonometry in with their usual studies. *Common Core* is rudderless! However, to do my part, I proved the first and second laws of sines and of cosines, though these really belong in the blue-belt chapter; they are about ratios, and the law of sines cites the inscribed angle theorem, which is green belt. Recognizing that there are many students here, at the end of green belt, that I will never see again – they will accept their C grades and make a run for it – I feel that it is my responsibility to fill green-belt students in on some more trigonometry.

$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$ High-school students are usually not expected to prove this identity. Combined with the second law of sines, $\frac{e-f}{e+f} = \frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}$, we get these laws:

First Law of Tangents $\frac{e-f}{e+f} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$

Second Law of Tangents $\frac{e}{f} = \frac{1+T}{1-T}$ with $T = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$

To solve a triangle means to be given three magnitudes and to find the other three. Yellow belts learn that this is possible if we are given SAS, SSS, ASA, or AAS; if given ASS, there are two possible triangles, a point often missed by algebra students who rely too heavily on their scientific calculator, because its $\text{asin}(\quad)$ button outputs only the acute angle. Many teachers have noted that these problems can always be solved with the laws of sines and of cosines, so the law of tangents seems unneeded. We will now use the law of tangents solve two real-life problems.

The church steeple is crooked! Even worse, it is leaning at an angle perpendicular to the view from the touristy part of town. The priest is getting *really* tired of tourists knocking on his door to tell him that his steeple is crooked, so he has decided to fix it. But, before renovations can begin, he must determine the exact skew angle, so it can be tilted to the vertical. The internal braces make it impossible to get inside the steeple to measure its base, or either base angle. But, on the outside, it is possible to measure the sides at 30.0 m and 30.1 m; the apex angle is 2°.

By the angle sum theorem, $\alpha + \beta = 180° - 2° = 178°$

By the first law of tangents, $\alpha - \beta = 2\,\text{atan}\left(\frac{30.1-30.0}{30.1+30.0}\tan\frac{178}{2}^{\circ}\right) \approx 10.89°$

Adding these together gives us $\alpha = 94.445°$ and $\beta = 83.555°$, and the law of sines gives us the base, 1.054 m, though only $\alpha - \beta$ was asked for. The advantages over the law of cosines are:

1. Subtracting the nearly equal 1804.90 from 1806.01 loses three digits of accuracy, which goes unnoticed on a 12-digit calculator, but it is a problem in QBasic with its 7 digits.

2. Evaluating the square root of 1.11 is easy today, but before calculators and computers, it was done iteratively. Guess at $y_0 = 1$. $y_1 = \frac{y_0 + \frac{1.11}{y_0}}{2} \approx 1.055$, $y_2 = \frac{y_1 + \frac{1.11}{y_1}}{2} \approx 1.0535664$, $y_3 = \frac{y_2 + \frac{1.11}{y_2}}{2} \approx 1.0535654$. One iteration is sufficient for carpentry, but it still takes time.

3. By ASS, $\alpha = \text{asin}\left(\frac{30.1}{\sqrt{1.11}} \sin 2°\right) \approx 85.610°$ or $180° - 85.610° \approx 94.390°$. Calculators output the acute angle; if you are not paying attention, you may not realize that you want the obtuse angle. Also, $94.445°$ is more accurate than $94.390°$.

In the middle of a severe winter, there is concern that the roof of the high-school gymnasium will collapse under the snow load. The roof is asymmetrical, with the southern side longer to support solar panels. It would be difficult to remove snow without damaging the solar panels, so the principal has hired a structural engineer to assess the strength of the roof to decide if snow removal is necessary. He replies that he can do this for a symmetrical roof and, if given the ratio of the long side to the short side, he can estimate how much weaker the asymmetrical design is.

Unfortunately, the blueprints for the building are lost. Also, while people can get under the roof near its apex, the edges are too narrow to crawl into, so it is impossible to measure the base or either leg of the triangle, nor to measure the base angles. Unsure of how to proceed, the principal asks the geometry teacher if he and his students can find this ratio, working only near the apex.

The students measure the apex angle, $\gamma = 166°$. By the angle sum theorem, $\alpha + \beta = 14°$. With a carpenter's square held against the north rafter and another against the south rafter, the students locate a point on the ceiling joist that is equidistant from each rafter; that is, the two carpenter's squares measure the same perpendicular distance to that point. By the angle bisector theorem, this point is G^*, the intersection of the apex angle bisector with the base. The students then suspend a plumb bob from the apex and observe that it is 2 meters long and 35 mm from G^*; so, $\text{atan}\frac{2.00}{.035} \approx 89°$. Thus, by the skew angle theorem, $\alpha - \beta \approx 91° - 89° \approx 2°$.

Let $T = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} = \frac{\tan(1°)}{\tan(7°)} \approx 0.142$. By the second law of tangents, $\frac{e}{f} = \frac{1+T}{1-T} \approx \frac{1.142}{0.858} \approx 1.331$.

Tangential Quadrilaterals Revisited

I put tangential quadrilaterals in yellow belt because the Pitot theorem and, thanks to Fetisov's criticism of Glagolev, its converse too, are neutral geometry. But textbooks that do not make a distinction between neutral and Euclidean geometry wait until the inscribed angle and cyclic quadrilateral theorems have been proven so problems like the following can be solved.

Red belts of *Geometry–Do* will not be working with tangential quadrilaterals, so this material would have interrupted the green belt's preparation for red belt, which proves some of the classic results of geometry that all intermediate geometers are expected to know. Thus, to avoid having the students go off on a tangent (no pun intended), I put these problems in an optional green-belt appendix. We will start with the easy ones!

Problem 4.39

Let $\overline{EFGH}$ be tangential but not a square with I its incenter, $I_{EF}, I_{FG}, I_{GH}, I_{HE}$ the incircle's touching points and P the intersection of $\overline{I_{EF}I_{GH}}$ and $\overline{I_{FG}I_{HE}}$. Draw a line through P perpendicular to $\overline{IP}$ and label its intersections with $\overline{HE}$ and $\overline{FG}$ as J and K, respectively. Prove $\overline{I_{HE}J} = \overline{I_{FG}K}$.

> ***Solution***
>
> $\overline{I_{HE}I} = \overline{I_{FG}I}$ and, by the tangent theorem, $\angle JI_{HE}I = \angle KI_{FG}I$. By the cyclic quadrilateral theorem converse, $\overline{IPJI_{HE}}$ is cyclic; by the inscribed angle theorem, $\angle I_{HE}IJ = \angle I_{HE}PJ$. By the inscribed angle theorem converse, $\overline{IPI_{FG}K}$ is cyclic; by the inscribed angle theorem, $\angle I_{FG}PK = \angle I_{FG}IK$. By the vertical angles theorem and transitivity, $\angle I_{HE}IJ = \angle I_{FG}IK$. By ASA, $\overline{I_{HE}IJ} \cong \overline{I_{FG}IK}$, which holds the equality $\overline{I_{HE}J} = \overline{I_{FG}K}$. ∎

Problem 4.40

Let $\overline{EFGH}$ be tangential with I its incenter. Draw lines through E, F, G, H perpendicular to $\overline{IE}, \overline{IF}, \overline{IG}, \overline{IH}$, respectively. Let J, K, L, M be the intersections of these lines that are long of $\angle EIF, \angle FIG, \angle GIH, \angle HIE$, respectively. Prove that I is the bi-medial of $\overline{JKLM}$.

> ***Solution***
>
> By the cyclic quadrilateral theorem converse, $\overline{IEJF}, \overline{IFKG}, \overline{IGLH}, \overline{IHME}$ are all cyclic.
>
> | | | |
> |---|---|---|
> | $\angle KJM + \angle JKI + \angle JMI$ | $= \angle KJI + \angle MJI + \angle JKI + \angle JMI$ | angle addition |
> | | $= \angle FEI + \angle EFI + \angle FGI + \angle EHI$ | inscribed angle th. |
> | | $= \frac{1}{2}(\angle E + \angle F + \angle G + \angle H)$ | definition of incenter |
> | | $= \sigma$ | quadrilateral angle sum th. |
>
> Thus, K, I, M are collinear. Analogously, J, I, L are collinear. ∎

These problems used only the definition of tangential quadrilaterals. Now let us see if you remember the theorems that we proved about them when you were a yellow belt.

Problem 4.41

Given cyclic quadrilateral $\overline{EFGH}$ such that $\angle E \neq \angle G$, let I and J be the incenters of $\overline{EGH}$ and $\overline{EFG}$, respectively. Prove that $\overline{EFGH}$ is tangential if and only if $\overline{IJFH}$ is cyclic.

Solution

Assume $\overline{EFGH}$ is tangential. By tangential quadrilateral theorem II, the incircles of $\overline{EGH}$ and $\overline{EFG}$ are tangent; let their touching point be P. Let J' be the foot of the perpendicular dropped from J to $\overline{FG}$. Let K be a point on $\overrightarrow{IJ}$ that is past J. By the cyclic quadrilateral theorem converse, $\overline{JJ'GP}$ is cyclic; thus, $\angle FGP = \sigma - \angle PJJ' = \sigma - (\sigma - \angle FJJ' \pm \angle KJF)$. Thus, $\angle KJF =$ [a series of equalities] $= \angle IHF$. By the cyclic quadrilateral theorem converse, $\overline{IJFH}$ is cyclic. •

The series of equalities leading to $\angle KJF = \angle IHF$ is left as an exercise. Also, proof that $\overline{IJFH}$ being cyclic implies that $\overline{EFGH}$ is tangential is left as an exercise; just walk the proof backwards.

If $\overline{EFGH}$ is a right kite with $\angle E = \angle G = \rho$, then $\overline{EFGH}$ is tangential but $\overline{IJFH}$ cannot be cyclic because I, J, F, H are collinear and collinear points do not define a circle. There is no such thing as a circle of infinite radius because infinity is not a point and circles have their centers at points.

If $\overline{EFGH}$ is bi-centric but differs only slightly from a right kite with $\angle E \approx \angle G \approx \rho$, then $\overline{IJFH}$ will be such a skinny quadrilateral that the center of its circumcircle is not just off the edge of the paper but into the next county.[92] A computerized check of whether the mediators concur (diameter and chord corollary #4) finds that they do not; it is unclear if this is due to numerical error or to $\overline{IJFH}$ not being cyclic. But, with the certainty that only deductive logic guarantees, a geometer can say, "If $\overline{EFGH}$ is bi-centric, then $\overline{IJFH}$ is cyclic. Always – no 'ifs,' 'ands' or 'buts.'"

Suppose you began geometry with a classmate who was/is a computer programmer. He got as far as the tangent theorem before he dropped out, loudly announcing that he despises logical arguments and, except for the theorems he learned so far, he will trust only in numerical results. Now he scoffs at your red-belt logic. Ask him if $\overline{IJFH}$ is cyclic when $E \equiv \left(-\frac{28561}{40}, \frac{28561}{30}\right)$, $F \equiv \left(\frac{2975}{4}, 750\right)$, $G \equiv \left(\frac{114244}{119}, -\frac{85683}{119}\right)$ and $H \equiv \left(-1000, -\frac{2975}{3}\right)$. *That should shut him up!*

[92] Province, oblast; whatever you call the interior regions in your country.

The Way Forward

In the *Note to Teachers* section at the beginning of this book, I write:

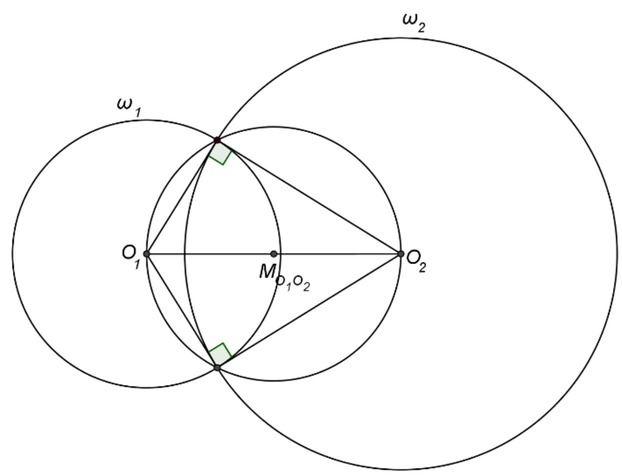
> *Red belt is needed for black belt, but not for blue belt. If you have no plans for a third year, you may consider skipping the red-belt chapter and studying blue-belt quadrature in the fourth semester.*

If this is the case, I would still recommend that students read the biographical sketches at the beginning of red belt and – especially for aspiring military officers – the elementary work of Auguste Miquel. After Miquel is the long circle theorem. In some textbooks, this is the crowning achievement of what they call elementary geometry; roughly, our green belt. So, read it too.

Orthogonal Circles Theorem

Given two overlapping circles, they are ***orthogonal*** *if and only if any of these conditions hold:*

1. *Radii of the two circles to an intersection point are perpendicular.*
2. *A radius of one circle to an intersection point is tangent to the other circle.*
3. *The circle whose diameter is from center to center passes through their intersections.*



Parts (1) and (2) are neutral geometry while (3) cites Thales' diameter theorem. Construction 4.23 is also neutral geometry, as is P. 4.42 for circles outside each other. Thus, orthogonal circles could have been defined for yellow belts, and some textbooks do define them this early. But most of the interesting problems cite the power of the point (blue belt), harmonic division (*Cho–Dan*) or circle inversion (*Yi–Dan*), which few yellow belts will get to. What is annoying about *Common Core* is its practice of defining terms and then never using them; so, to avoid being like that and because yellow belts already have a lot on their plate, I waited to define orthogonal.

Construction 4.23 *Given a circle and a point, construct an orthogonal circle through the point.*

Problem 4.42 *Given three circles with three touching points, prove that the circle through the three touching points is orthogonal to all three given circles.*

Problem 4.43 *Given two orthogonal circles, prove that the two lines from their two intersections to a point on one circle meet the other circle at diametrically opposite points.*

Problem 4.44 *Given $\overline{EFGH}$ cyclic with $\overline{EF}$ a diameter and T the bi-medial point, prove that a circle with common chord $\overline{GH}$ is orthogonal if and only if it passes through T.*

Problem 4.45 *Given $\overline{EF}$, a diameter of ω, and G any point on ω, prove that the circles through E, M_{EF}, G and through F, M_{EF}, G are orthogonal.*

The next two problems cite the long circle theorem, which is red belt; but, the corollary can be proven independently, without any mention of the excenter.

Long Circle Theorem Corollary
Given $\overline{EFG}$ with circumcircle ω, then E, F and I are equidistant from the ***long center****, L_G.*

Problem 4.46 *Tangents to a circle at E, F meet at G; prove that the $\overline{EFG}$ incenter is on the circle.*

Problem 4.47 *Given $\overline{EFG}$ with incenter I and excenters X, Y, Z, prove that the circles with diameters $\overline{IX}$ and $\overline{YZ}$ are orthogonal.*

The Euler circle is definitely advanced triangle geometry – I am not aware of any elementary textbook that discusses it – but it is not *very* advanced, usually appearing in Chapter One. This is true of *The Geometry of Remarkable Elements* by Constantin Mihalescu, who devotes his first chapter to the subject, an incredible 119 pages! Red belts learn of it, but not quite that deeply. Problem 4.48 is a trick question because it does not say "Euler circle." It is hard but doable now.

Problem 4.48 *Given $\overline{EFG}$ with orthocenter H, prove that the circles with diameters $\overline{EH}$ and $\overline{FG}$ are orthogonal.*

The orthogonal lens area theorem is trigonometry; Mathematical Olympians should know it.

Orthogonal Lens Area Theorem ***216***
The overlap of orthogonal circles with radii R and r has area $A = (R^2 - r^2)\,\mathrm{atan}\frac{r}{R} + \frac{\pi}{2}r^2 - Rr$ or $r^2\left(\frac{\pi}{2} - 1\right)$ if $R = r$. For general but equal circles, $A = r^2(\theta - \sin\theta)$ for θ not necessarily $\frac{\pi}{2}$.

Red Belt Instruction: Famous Theorems

This chapter is about geometry theorems difficult enough that they went unsolved for decades and are now named after famous mathematicians. Let us learn a little about these men!

Auguste Miquel (France, 1816 – 1851) was doing original research in geometry at a time when he should have been studying for the *Grandes Ecoles*, which is roughly equivalent to modern America's *Graduate Record Exam* (GRE). Doing well is required to get into graduate school. Miquel failed the exam and became a *régent*, which qualified him to teach in high school; he never became an *agrégé*, which is needed to teach in college. At twenty, he had already published in the journal *Le Géomètre* and, undaunted by his failure to become an *agrégé*, he published a series of articles in the prestigious *Journal de Mathématiques Pures et Appliquées*.

Miquel's story should be inspiring to students who know that their family cannot afford to send them to college even for a B.S., much less a Ph.D. Geometry is the one field where it is possible for people with little formal education to make a name for themselves. *Geometry–Do* does not even assume a knowledge of multiplication, much less of calculus. There is no other science that does not prerequisite multi-variable calculus and, as a practical matter, one cannot get far in engineering without a working knowledge of differential equations. While it is not impossible for an autodidact to learn calculus, geometry is the one field where a man with no formal education can suddenly appear on the world stage with a theorem that has everybody on their feet.

Miquel's story should also be inspiring to Americans with a B.S. in mathematics who discover, to their chagrin, that it is a worthless degree. High schools require a B.S. in education, universities require a M.S. in mathematics and tech companies require a B.S. in engineering. Exactly nobody accepts job applicants with a B.S. in mathematics. What actually becomes of them is that they are hired as substitute teachers, the same job that is normally given to housewives, except that they do it as a part-time job, not just in rare cases when a teacher gets sick, like the housewives.

Sadi Carnot (France, 1796 – 1832) was a retired army officer when France was in disarray after the fall of Napoleon. Steam power was used in France, but only inefficiently and for primitive tasks such as draining mines or grinding grain. Carnot was impressed that British steam power had become much more advanced, and mostly through the insights of a few engineers who had no formal training. Their work was empirical; that is, while they took careful measurements of operational steam engines, their data was interpreted mostly with the aid of intuition. Carnot was convinced that the inadequacy of France in this regard was a large part of its downfall and that, if only the rigor of geometry could be applied to this practical science, efficient use of steam power could propel France once again to greatness. Instead of studying the minutia of real steam engines, as the British did, Carnot devised a set of postulates to describe what is meant by an abstract steam engine, just as Euclid devised a set of postulates to describe geometry. His work, published under the title *Reflections on the Motive Power of Fire*, today earns him the exalted title of being the Father of Thermodynamics. We owe to Carnot the realization that, while careful measurements are important, they must be guided by deductive logic, not just the vague hope that intuition will somehow see the path to an improved steam engine in all that data.

William Wallace (Scotland, 1768 – 1843) was named after the famous Scottish general, but lived 500 years later and, while he would become a professor at the Royal Military College, he was not a military man. The son of a leather worker, he had no formal education after the age of eleven. While earning his living as a book seller and as a private math tutor, he came under the wing of John Playfair, whose version of Euclid's parallel postulate we employ in *Geometry–Do*. With Playfair's endorsement, Wallace became a professor at the Royal Military College. In 1819, he achieved the chairmanship at Edinburgh, the most exalted position for a Scottish mathematician.

Wallace's story should be inspiring to people whose path is blocked by those who would get ahead by currying favor with powerful but uninformed politicians. Wallace had the support of respected mathematicians such as Playfair, while his rival had the support of powerful politicians, but it was Wallace who was elevated to the mathematical chair of Edinburgh. This is a remarkable achievement for someone who dropped out of school at the age of eleven. He held this chair for twenty years and is remembered for taking the winning (Continental) side in the dispute over who invented the calculus and whose notation would be used. He also invented the pantograph.

Evangelista Torricelli (Italy, 1608 – 1647) was invited to Florence to serve as Galileo's assistant in the last three months of Galileo's life; he then succeeded him as professor of mathematics at the Florentine Academy. Galileo is famous and his contributions to science – indeed he founded what science means to us today – are voluminous. Whole books can and have been written about him, so we will pause here only to note that it was a lecture on geometry that turned Galileo away from the study of medicine, which his father had encouraged because doctors make more money than mathematicians. But, despite Galileo's fame, the torch he lit could have easily flickered out had it not been for a series of brilliant Italian geometers who followed in his path.[93]

Galileo was an astronomer who supported – and got in trouble for doing so – heliocentrism, the belief that the Earth rotates around the sun, not the sun around the Earth. Also, he famously dropped two spheres of different masses from the leaning tower of Pisa to demonstrate that they strike the ground simultaneously. But he was aware that this experiment does not exactly

[93] Those mentioned in this book are Torricelli, Viviani, Ceva, Saccheri and Fagnano, in this order; there were others.

work. We now know that the ballistic coefficient of a sphere – how well it slips through the air – is proportional to its mass and inversely proportional to the square of its diameter. The volume of a sphere is $V = \frac{4}{3}\pi r^3$, so its ballistic coefficient is $c\frac{r^3}{r^2} = cr$, with c a constant determined by the surface roughness. Thus, the bigger ball hit first, though the difference is negligible in the short fall from the Tower of Pisa. The effect is more noticeable for balls fired at an angle; Galileo's theory predicts a parabolic path, yet cannon balls fall far short of this. Galileo knew that this is due to air resistance, but he had no theory of what air is; he did not know that it has weight.

It was Galileo's student Torricelli who famously inverted a 1.2 m test tube of mercury in a dish of mercury, measured the height of the liquid and concluded that it was the weight of the air in the atmosphere pressing down on the mercury that prevented the tube from draining completely so the mercury inside it and the mercury outside it would be level. Air is like a very thin liquid! Thus, Torricelli invented the barometer and is today known as the Father of Hydrodynamics.

Leonhard Euler (Switzerland, 1707 – 1783) was every bit as prolific as Galileo; I cannot begin to here list his many accomplishments. I will note only that his contribution to ballistics was analogous to that of Carnot's contribution to thermodynamics. Artillerists of the time were wont to take very careful measurements of the range of guns; but, because extant guns were never elevated above 5° due to nobody understanding ballistics well enough to attempt higher angles of fire, the ball skimmed just above the ground. Slight variations in the powder charge or gradual slopes caused the ball to strike many meters before or after its expected impact. The result was that all those careful measurements were only sowing confusion among the gunners.

Euler understood that, if ballistics was to be a real science, it must employ the methodology used by Euclid in geometry; that is, Euler had to devise a set of postulates to describe an abstract gun.

1. Constant atmospheric density from the ground to the apogee.
2. Drag is everywhere proportional to the square of the speed.
3. Gravity is everywhere pointed downwards; *i.e.* the Earth is flat.

This results in an unsolvable differential equation, though it can be approximated with the Runga-Kutta method. Euler was aware that Torricelli had taken his barometer up mountains and found that the air pressure varies inversely with altitude, but he deemed this immaterial because guns of his day were not powerful enough to fire shells this high. And, of course, he knew that the Earth is round, but he deemed this immaterial because guns of his day were not powerful enough to fire shells over the horizon. Euler's second postulate is only true up to 240 m/s, which Euler was probably not aware of because guns of his day did not have even half this muzzle velocity. Thus, in point of fact, none of Euler's three postulates paint a complete picture of the situation.

The important lesson that modern geometers can take from Euler's work is that they must ignore the naysayers who criticize their postulates as too abstract. Euler understood that his postulates describe a simple case that does not quite exist in reality; but he also understood that, once he had figured out ballistics for this simple case, it would be easy to extend it to guns that fire high enough to reach thin air and far enough to hit ships off the coast for whom only the tips of their masts are visible. Smokeless powder was not invented in his lifetime; but, when it was, artillerists easily extended Euler's theory to projectiles over 240 m/s; drag is proportional to the cube of velocity. Any modern artillerist, steeped in the blizzard of differential equations that ballistics has become, still recognizes Euler's three postulates as the foundation of all that he studies.

Sadly, the siren song of "data" is not easy to ignore. Those who make a fetish out of measuring things scorned Euler and, as late as the U.S. Civil War, were still firing their guns in increments of $0°, 1°, 2°, 3°, 4°$ and $5°$, walking out in the dirt to look for scrapes where the ball first touched down, and then assembling a jumble of statistics that served only to confuse gunners. It is not an exaggeration to say that, despite Tredegar Iron Works being their only foundry, had the Rebels heeded Euler from a hundred years in their past, they might have won that war. It is absurd that they went to war with nothing for fire control beyond a jumble of incoherent statistics that could not even be used to interpolate between $0°$ to $5°$, much less to raise the barrel to higher angles.

Whataboutists have a hundred objections to any axiomatic theory: What about air getting thinner at high altitude? What about the curvature of the Earth? What about humidity? What about the gravitational tug of Mars? They have no answers to these questions; they ask them only to throw hedgehogs at the feet of the axiomatist. Then they go back to the compiling of statistics, which is all that they know. Even without answering such obstructionist questions, use of Euler's theory from three klicks away would have sure had those Yanks jumping and stepping!

At the end of white belt, there is a section titled *Defense Positioning and Geometry*, which is based on the first chapter of Raj Gupta's book of the same title (2003). Raj Gupta is a quantitative analyst – as is Xing Zhou, the author of a problem book mentioned in the references – but Gupta is also the coauthor of a book, *Controlling the Greenhouse Effect: Five Global Regimes Compared*. So, Mr. Gupta knows a thing or two about weather prediction.

In sharp contrast, Mark Buchanan is an "economist" who knows nothing about quantitative analysis and even less about weather forecasting, which makes it awfully nervy of him to title his 2013 economics book, *Forecast*. I wrote a short four-page review[94], which is well within the abilities of high-school geometers to read. I highly recommend that they do! I posted this paper at *Research Gate* in 2013 and today, 2020, it has only gotten 217 reads. This is probably because

[94] www.researchgate.net/publication/270821186_Review_of_Mark_Buchanan's_Book_Forecast

debunking mountebanks is seen as fruitless – like throwing pies at a clown; most of them miss and the ones that do hit just invoke giggles from the clown – but geometers are logicians and, as such, they must understand how and why the enemies of logic always manage to bounce back.

Because this four-page paper is so easy to read, I will not quote from it here. Suffice it to say that Mark Buchanan is left with my footprint on his chest and high-school students – especially those who read the fifth page that summarizes Lewis Fry Richardson's 1922 book *Weather Prediction by Numerical Process* – will be left with a summary knowledge of the axiomatic foundation of weather prediction. This fifth page should make it clear that Richardson bases his seven axioms on an even older axiomatic system, Daniel Bernoulli's 1738 *Hydrodynamica*. Like Leonhard Euler, whose axiomatic theory of ballistics lay dormant until WWI, when the invention of smokeless powder finally made it imperative that artillerists stop it with the stupid gathering of data and learn some real theory, Bernoulli was also ahead of his time. It would be 165 years before the Wright Flyer took wing at Kitty Hawk, but it was Bernoulli that the Wright Brothers were reading.

I will here quote at length from Raj Gupta's *Defense Positioning and Geometry* to make it clear that Mr. Gupta too understands that all real scientific advances begin with the axiomatic method.

> *The flow of fluids in the Earth's atmosphere is extremely complicated. In the language of physics, it is turbulent, rotational, compressible and viscous. No foreseeable advance in theory or computational ability will* ever *enable us to model or predict all the complexities of fluid flow in the real atmosphere. Yet, despite this seemingly insurmountable obstacle, weather prediction has evolved into a very reliable and precise science over the last few decades. One way to attack the problem of weather prediction is to start with a very simplified and streamlined conceptual model of a fluid in motion and examine how such a fluid should behave. This was the approach adopted by Daniel Bernoulli (1700–82) in formulating his basic equation describing the pressure, velocity, and density of an idealized incompressible and nonviscous fluid in steady motion. Bernoulli's Law, which is one of the basic principles of fluid dynamics, was first presented in Bernoulli's* Hydrodynamica *in 1738.*

Starting with a "very simplified and streamline conceptual model" is exactly the opposite of Jean-Philippe Bouchard's "statistical regularities should emerge," as cited by Mark Buchanan:

> *The goal [of economics] is to describe the behavior of large populations, for which statistical regularities should emerge, just as the law of ideal gasses emerge from the incredibly chaotic motion of individual molecules.*

Bouchaud is absurd: a statistical result requires a datum for every individual element. Yet we do not have data on every individual molecule, or even on one of them. They are too small to see. But let us be positive and quote some more from Raj Gupta, *Defense Positioning and Geometry*.

> *This book is about the geometry and density of conventional forces especially at low force levels. It attempts to do for conventional defense and conventional warfare what the equations of fluid flow did for weather prediction. In other words, this book lays out the laws of force positioning and concentration that all defending force structures must abide by to field an effective defense. This book uncovers certain special relationships that contribute to a successful defense and shows how they can be exploited to construct stable military balances and to reduce forces to minimum levels. Of course, predicting the precise outcome of a war between two existing power blocs is akin to predicting the day-to-day weather in New York City or the likelihood of tornadoes in Oklahoma. The level of discussion in this book is far more basic – closer to the equations of fluid dynamics than to the weather prediction models that are founded on these equations. Thus, in terms of weather prediction, the first step must be to understand how an idealized fluid in motion behaves and what factors determine its properties… In terms of military defense, this means without first understanding how a model defense should be structured, there is little hope of* confidently *ascertaining whether a real-world defense is optimal… To derive the fundamental rules of force positioning one has no choice but to abstract from the tremendous complexities of real battlefields and war situations (p. 1-3).*

This is how real science works. When someone says, "statistical regularities should emerge," you know they are a clueless idiot engaged in blind guesswork; their statistics are a record of failure.

Miquel Theorem

Given $\overline{EFG}$ and arbitrary points J, K, L on $\overline{EF}$, $\overline{FG}$, $\overline{GE}$, respectively, the circumcircles of $\overline{EJL}$, $\overline{FKJ}$ and $\overline{GLK}$ are concurrent. The ***Miquel circles*** *are $\omega_E, \omega_F, \omega_G$ with centers O_E, O_F, O_G, respectively.*

> ***Proof***
>
> The **Miquel point**, M, is the intersection of ω_E and ω_F that is not J. Assume M is inside the triangle. $\overline{EJML}$ and $\overline{FKMJ}$ are cyclic. By the cyclic quadrilateral theorem, $\angle EJM$ and $\angle MLE$ are supplementary, as are $\angle MJF$ and $\angle FKM$. $\angle EJM$ and $\angle MJF$ are supplements, so $\angle MLE$ and $\angle FKM$ are also. Thus, their supplements are supplementary, $\angle GLM$ and $\angle MKG$, respectively. By the cyclic quadrilateral theorem converse, $\overline{GLMK}$ is cyclic. •

M outside the triangle is left as an exercise. This theorem is also true for J, K, L on $\overleftrightarrow{EF}, \overleftrightarrow{FG}, \overleftrightarrow{GE}$.

We will use the notation established in the Miquel theorem without further explanation. We will here assume that M is inside the triangle. When we learn of the Wallace line, the point on the circumcircle is the Miquel point. There are other theorems, but red belts will treat Miquel lightly.

Miquel Equal Angle Theorem
Lines from the Miquel point to the Miquel circle intersections make equal angles with the sides.

Proof
$\angle MJE$ and $\angle MJF$ are supplementary because they are on one side of $\overleftrightarrow{EF}$. $\angle MJF$ and $\angle MKF$ are supplementary by the cyclic quadrilateral theorem. Thus, $\angle MJE = \angle MKF$. Analogously, $\angle MJE = \angle MKF = \angle MLG$. ■

Reverse Miquel Construction
Given M inside $\overline{EFG}$, find J, K, L on $\overline{EF}, \overline{FG}, \overline{GE}$, respectively, such that M is the Miquel point.

Solution
Choose J on $\overline{EF}$ arbitrarily. By C. 3.4, find K and L such that $\angle MJE = \angle MKF = \angle MLG$. Since J is arbitrary, there are an infinity of solutions; the construction is under defined. ■

Problem 5.1 *If three circles overlap in pairs, prove that their common chords are concurrent.*

Farmers cut fields out of wilderness in the shape of rectangles because it is easiest to drive their tractors back and forth across a rectangle than any other polygon. But soldiers cut bases out of enemy territory in the shape of triangles because they are only given three automatic cannons to defend them with. Four would be nice, but cannons are not cheap, so they usually have three. Suppose that some impertinent enemy captain has built such a base in your land, and you wish to put an end to this nonsense. There are three requirements that the initial bombing must have:

1. Every part of the triangle must be struck by shrapnel from at least one bomb.
2. An important point inside the triangle must be struck by shrapnel from every bomb.
3. The bombs are of the same type and thus they have equal-size circles of shrapnel.

First, we hit the important point with a bunker-buster bomb; but this has little effect on nearby troops. If a target can be destroyed with three bombs, then it is the Marine Corp way to destroy it with ten! However, if Marines are poised to overrun the base after the enemy is stunned by your bombing, then you do not want to overdo it lest you harm friendly units or civilians nearby.

What is the minimal amount of ordnance needed to accomplish the three requirements?

The reverse Miquel construction satisfies the first two requirements. Since J is chosen arbitrarily, it might be hoped that a choice exists that satisfies the third requirement. No; it depends on M.

Equal Miquel Circles Theorem
The Miquel circles are equal if and only if the Miquel point is at the circumcenter of the triangle.

Proof
Assume that the Miquel circles are equal. By the Miquel equal angle and the inscribed angle theorems, $\angle EO_EM = \angle FO_FM = \angle GO_GM$. By SAS, $\overline{EO_EM} \cong \overline{FO_FM} \cong \overline{GO_GM}$, which holds the equality $\overline{EM} = \overline{FM} = \overline{GM}$. •

The reverse implication is left as an exercise. This proves that satisfying the third condition is in the hands of the enemy, since they are the ones who drew $\overline{EFG}$ and located M inside it. But the incenter and circumcenter theorem shows that an equilateral triangle with an important point at the center (*e.g.,* a munitions dump) is the best defense against both enemy aircraft and enemy troops if you have only three anti-aircraft guns. This motivated the Dakota defense problem.

Dakota Attack Problem
Bomb an equilateral triangle with three equal-size bombs so every part is struck by shrapnel from at least one bomb and the incenter/circumcenter is struck by shrapnel from every bomb.

Solution
Use the reverse Miquel construction with the Miquel point at the circumcenter. J on $\overline{EF}$ such that $\angle MJE = \rho$ is the obvious choice, but any angle will work and angling it towards the troops poised to overrun the enemy base puts them between two bombs. ■

The solution to the Dakota attack problem gives the minimal amount of ordnance needed to accomplish the three requirements when bombing an equilateral triangle, but you will probably want to choose the next larger size of bomb – they come in discrete sizes – to overlap the vertices and the important point a little. If the enemy has drawn a sloppy equilateral triangle, but your bombs overlap the circumcenter enough to reach the important point, then you are good.

Miquel Similarity Theorem
The centers of the Miquel circles are vertices of a triangle similar to the given triangle.

The points where the bisectors of the interior angles of a triangle, $\overline{EFG}$, intersect the circumcircle, L_E, L_F, L_G, are called **long centers** because, by the following theorem, they center the **long circles**, $\omega_E, \omega_F, \omega_G$. They are vertices of the **long triangle**, $\overline{L_EL_FL_G}$; it has the same circumcircle as $\overline{EFG}$.

Long Circle Theorem
Given $\overline{EFG}$ with circumcircle ω, then I, E, X, F are concyclic and their center is L_G.

Proof
By the interior and exterior angles theorem, $\angle IEX$ and $\angle IFX$ are right. By the cyclic quadrilateral theorem converse, I, E, X, F are concyclic and their center is the midpoint of $\overline{IX}$. We will call this circle $\omega_{M_{IX}}$ with center M_{IX}. I and X are both on the bisector of $\angle G$, so $\angle GM_{IX}F = \angle IM_{IX}F$. By the inscribed angle theorem in $\omega_{M_{IX}}$, $\angle IM_{IX}F = 2\angle IEF$. But, by the incenter theorem, $2\angle IEF = \angle GEF$. Thus, $\angle GM_{IX}F = \angle GEF$ and, by the inscribed angle theorem converse, M_{IX} is on ω. By the mediator and angle bisector theorem, $\overleftrightarrow{GI}$ intersects ω only at G and L_G. M_{IX} is not G, so it must be L_G. ■

The long circles, $\omega_E, \omega_F, \omega_G$, are Miquel circles and the incenter, I, is the Miquel point of the **double-long triangle**, $\overline{XYZ}$, with E, F, G each on a different side. $\overline{XYZ} \sim \overline{L_E L_F L_G}$, by the Miquel similarity theorem, as is any $\overline{X''Y''Z''}$ with X'', Y'', Z'' on the long circles and E, F, G each on a different side. $\overline{XYZ} \sim \overline{X''Y''Z''}$ have the same Miquel point, I, but are of different size and tilt.

Largest Reverse Miquel Triangle Theorem
For J, K, L and Miquel point M, the largest $\overline{EFG}$ such that $J \in \overline{EF}, K \in \overline{FG}$ and $L \in \overline{GE}$ is the one for which $\overline{EF} \perp \overline{MJ}$, $\overline{FG} \perp \overline{MK}$ and $\overline{GE} \perp \overline{ML}$.

Long Triangle Theorem
The incenter of a triangle is the orthocenter of its long triangle.

Proof
Given $\overline{EFG}$ with incenter I and circumcenter O and its long triangle $\overline{L_E L_F L_G}$, let $P := \overline{L_E L_G} \cap \overline{FL_F}$. By the intersecting chords angle theorem,
$\angle FPL_G = \frac{1}{2}(\angle FOL_G + \angle L_E OL_F) = \frac{1}{2}(\angle FOL_G + \angle L_E OG + \angle GOL_F)$
$\angle FPL_E = \frac{1}{2}(\angle FOL_E + \angle L_F OL_G) = \frac{1}{2}(\angle FOL_E + \angle L_F OE + \angle EOL_G)$
$\angle FOL_E = \angle L_E OG$ and $\angle L_F OE = \angle GOL_F$ and $\angle EOL_G = \angle FOL_G$ by the inscribed angle theorem, so $\angle FPL_G = \angle FPL_E$. These are supplements, so $\angle P$ is right; that is, $\overline{L_F P}$ is an altitude of $\overline{L_E L_F L_G}$ and, because $\overrightarrow{FL_F}$ bisects $\angle F$, it passes through I. Analogously, all the altitudes of $\overline{L_E L_F L_G}$ pass through I, so it is the orthocenter of $\overline{L_E L_F L_G}$. ■

Carnot Theorem
The sum of the perpendiculars dropped from the circumcenter onto the three sides of a not obtuse triangle is equal to the circumradius plus the inradius.

Proof

For right triangles, this is the right triangle incircle theorem. Given $\overline{EFG}$ acute, construct a right triangle with hypotenuse $\overline{IX}$ and legs parallel to $\overleftrightarrow{EF}$ and to $\overleftrightarrow{OL_G}$. The latter leg is $r_X + r$ long. By the long circle theorem corollary, M_{EF} is on $\overleftrightarrow{OL_G}$ and $\overleftrightarrow{OL_G} \perp \overleftrightarrow{EF}$ and hence parallel to this latter leg. Let M be the intersection of $\overleftrightarrow{OL_G}$ with the other leg. By the long circle theorem, L_G is the midpoint of $\overline{IX}$. By mid-segment theorem #2, M is a midpoint. By mid-segment theorem #1, $\overline{ML_G} = \frac{1}{2}(r_X + r\,)$. $\overline{M_{EF}L_G} = \overline{ML_G} - r = \frac{1}{2}(r_X - r\,)$. $\overline{OL_G} = R$, the circumradius, so $\overline{OM_{EF}} = R - \overline{M_{EF}L_G} = R - \frac{1}{2}(r_X - r\,)$. Thus,

$\overline{OM_{EF}} = R - \frac{1}{2}(r_X - r\,)$	
$\overline{OM_{GE}} = R - \frac{1}{2}(r_Y - r\,)$	Analogously for L_F
$\overline{OM_{FG}} = R - \frac{1}{2}(r_Z - r\,)$	Analogously for L_E
$sum = 3R - \frac{1}{2}(r_X + r_Y + r_Z) + \frac{3}{2}r$	Addition
$sum = 3R - \frac{1}{2}(r + 4R) + \frac{3}{2}r$	Excircle theorem corollary #1
$sum = R + r$	Simplify ■

By the mediator and angle bisector theorem, if L_{EF} is the intersection of the mediator of side $\overline{EF}$ of cyclic $\overline{EFGH}$ with the near arc of its circumcircle, then L_{EF} is on the angle bisector of both far vertices of $\overline{EFG}$ and $\overline{EFH}$; L_{EF} bisects $\angle EGF$ and $\angle EHF$. Analogously for L_{FG}, L_{GH} and L_{HE}.

Long Quadrilateral Theorem

*Given $\overline{EFGH}$ cyclic, the **long quadrilateral,** $\overline{L_{EF}L_{FG}L_{GH}L_{HE}}$, is orthodiagonal.*

Proof

Let O be the circumcenter of $\overline{EFGH}$ and T be the bi-medial of $\overline{L_{EF}L_{FG}L_{GH}L_{HE}}$. By the inscribed angle theorem, $\angle L_{HE}L_{EF}H = \angle L_{HE}L_{FG}H$ and $\angle L_{EF}L_{HE}F = \angle L_{EF}L_{GH}F$.

$\angle L_{HE}L_{EF}L_{GH} = \angle L_{HE}L_{EF}H + \angle L_{GH}L_{EF}H$	Addition
$= \angle L_{HE}L_{FG}H + \angle L_{GH}L_{EF}H$	Substitution
$= \frac{1}{2}\angle EL_{FG}H + \frac{1}{2}\angle GL_{EF}H$	Definition of L_{HE} and L_{GH}
$\angle L_{EF}L_{HE}L_{FG} = \angle L_{EF}L_{HE}F + \angle L_{FG}L_{HE}F$	Addition
$= \angle L_{EF}L_{GH}F + \angle L_{FG}L_{HE}F$	Substitution
$= \frac{1}{2}\angle EL_{GH}F + \frac{1}{2}\angle GL_{HE}F$	Definition of L_{EF} and L_{FG}
$\angle L_{HE}TL_{EF} = \sigma - (\angle L_{HE}L_{EF}L_{GH} + \angle L_{EF}L_{HE}L_{FG})$	Angle sum theorem
$= \sigma - \frac{1}{2}(\angle EL_{FG}H + \angle GL_{EF}H + \angle EL_{GH}F + \angle GL_{HE}F)$	Substitution
$= \sigma - \frac{1}{2}\sigma = \rho$	Inscribed angle theorem ■

Long Rhombus Theorem

Given $\overline{EFG}$ with incenter I and long centers L_E, L_F, L_G, let $J := \overline{L_E L_F} \cap \overline{GE}$ and $K := \overline{L_E L_F} \cap \overline{GF}$. Then, $\overline{GJIK}$ is a rhombus.

Proof

By the long triangle theorem, $\overline{JIKG}$ is orthodiagonal; call its bi-medial T. $\overline{JTG} \cong \overline{KTG}$ by ASA, so T is the midpoint of $\overline{JK}$. By the long circle theorem, I and G are on the long circle, so radii $\overline{L_F I} = \overline{L_F G}$. By the mediator theorem, T is the midpoint of $\overline{IG}$. By lemma 3.2.2, $\overline{GJIK}$ is a rhombus. ■

Cyclic/Tangential Pairs Theorem

*A quadrilateral is cyclic if and only if the **pedal quadrilateral** of its bi-medial point is tangential.*

Proof

Assume that $\overline{EFGH}$ is cyclic, T is its bi-medial and $\overline{T_{EF} T_{FG} T_{GH} T_{HE}}$ is its pedal quadrilateral. By the cyclic quadrilateral theorem converse, $\overline{T_{EF} F T_{FG} T}$ and $\overline{T_{GH} G T_{FG} T}$ are cyclic. By the inscribed angle theorem, $\angle T_{EF} F T = \angle T_{EF} T_{FG} T$ and $\angle T_{GH} G T = \angle T_{GH} T_{FG} T$. But, in the circumcircle of $\overline{EFGH}$, $\angle T_{EF} F T = \angle T_{GH} G T$, by the inscribed angle theorem. Thus, $\angle T_{EF} T_{FG} T = \angle T_{GH} T_{FG} T$, so T is on the angle bisector of $\angle T_{EF} T_{FG} T_{GH}$. Analogously, for $\angle T_{FG} T_{GH} T_{HE}$ and $\angle T_{GH} T_{HE} T_{EF}$ and $\angle T_{HE} T_{EF} T_{FG}$. Thus, by tangential quadrilateral theorem I, $\overline{T_{EF} T_{FG} T_{GH} T_{HE}}$ is tangential and T, the bi-medial of $\overline{EFGH}$ is its incenter. We do not call this incenter I because I is the incenter of $\overline{EFGH}$. •

Proof of the converse is left as an exercise.

Thus, cyclic and tangential quadrilaterals come in pairs; for every cyclic quadrilateral, the pedal quadrilateral of its bi-medial point is tangential. Might these pedal quadrilaterals also be cyclic? Yes, but only if the cyclic quadrilaterals that they are associated with are orthodiagonal; that is, their diagonals are perpendicular. We must have $\angle T = \rho$, where T is the bi-medial of $\overline{EFGH}$.

Lemma 5.1

The medial (Varignon) parallelogram of an orthodiagonal quadrilateral is a rectangle.

Proof

By mid-segment theorem #1, each medial parallelogram side is parallel to a diagonal. By the Lambert theorem corollary, the medial parallelogram is a right rectangle. ■

Cyclic and Orthodiagonal Theorem

A cyclic quadrilateral is orthodiagonal iff the pedal quadrilateral of its bi-medial point is cyclic.

Proof

Assume that $\overline{EFGH}$ is cyclic and orthodiagonal; $\angle T = \rho$. By Lemma 5.1, $\overline{M_{EF}M_{FG}M_{GH}M_{HE}}$ is a rectangle and $\overline{M_{FG}M_{HE}}$ is a diameter with its midpoint, M, the center of ω, its circumcircle. By Brahmagupta's bi-medial theorem, $\overline{M_{FG}T_{FG}M_{HE}}$ and $\overline{M_{FG}T_{HE}M_{HE}}$ are right triangles; by Thales' diameter theorem, T_{FG} and T_{HE} are on ω. Analogously, T_{EF} and T_{GH} are on ω. Thus, $\overline{T_{EF}T_{FG}T_{GH}T_{HE}}$ is cyclic, in the same circle as $\overline{M_{EF}M_{FG}M_{GH}M_{HE}}$.

Assume that $\overline{T_{EF}T_{FG}T_{GH}T_{HE}}$ is cyclic. By the cyclic quadrilateral theorem converse, $\overline{T_{FG}GT_{GH}T}$ and $\overline{T_{EF}FT_{FG}T}$ and $\overline{T_{HE}ET_{EF}T}$ and $\overline{T_{GH}HT_{HE}T}$ are cyclic: opposite right angles. Each of these circles are cited by the inscribed angle theorem in line four, below.

$\sigma = \angle T_{HE}T_{EF}T_{FG} + \angle T_{HE}T_{GH}T_{FG}$	Cyclic quadrilateral th.
$\quad = (\angle T_{FG}T_{EF}T + \angle TT_{EF}T_{HE}) + (\angle T_{FG}T_{GH}T + \angle TT_{GH}T_{HE})$	Addition
$\quad = (\angle T_{FG}T_{GH}T + \angle T_{FG}T_{EF}T) + (\angle TT_{EF}T_{HE} + \angle TT_{GH}T_{HE})$	Rearrange
$\quad = (\angle T_{FG}GT + \angle T_{FG}FT) + (\angle TET_{HE} + \angle THT_{HE})$	Inscribed angle th.
$\quad = \angle ETF + \angle ETF$	Exterior angle th.
$\rho = \angle ETF$	Halve both sides

Thus, $\overline{EFGH}$ is orthodiagonal. ■

We now consider a different pedal point, I, the incenter of $\overline{EFGH}$. $\overline{I_{EF}I_{FG}I_{GH}I_{HE}}$ is the **contact quadrilateral** of the incircle. P is its bi-medial, the intersection of its diagonals.

Bi–Centric Quadrilateral Theorem

A tangential quadrilateral is cyclic and thus bi-centric iff its contact quadrilateral is orthodiagonal.

Proof

Assume that $\overline{EFGH}$ is cyclic and thus bi-centric. $\overline{II_{EF}} \perp \overline{EF}$, $\overline{II_{FG}} \perp \overline{FG}$, $\overline{II_{GH}} \perp \overline{GH}$, $\overline{II_{HE}} \perp \overline{HE}$ by the tangent theorem, where I is the incenter of $\overline{EFGH}$. Thus,

$\angle I_{EF}PI_{HE} = \frac{1}{2}(\angle I_{EF}II_{HE} + \angle I_{FG}II_{GH})$	Intersecting chords angle theorem
$\quad = \frac{1}{2}\big((\sigma - \angle E) + (\sigma - \angle G)\big)$	Right cyclic theorem
$\quad = \sigma - \frac{1}{2}(\angle E + \angle G)$	Simplify
$\quad = \sigma - \frac{1}{2}\sigma = \rho$	Cyclic quadrilateral theorem

Assume that $\angle P = \rho$. By the inscribed angle theorem, $\angle I_{FG}I_{EF}I_{GH} = \angle I_{FG}I_{HE}I_{GH}$; call this α. By the tangent and chord theorem, $\alpha = \angle I_{GH}I_{FG}G$ and $\alpha = \angle I_{FG}I_{GH}G$. Analogously, $\beta = \angle I_{EF}I_{FG}I_{HE} = \angle I_{EF}I_{GH}I_{HE}$ and $\beta = \angle I_{HE}I_{EF}E = \angle I_{EF}I_{HE}E$. By the isosceles triangle theorem converse, $\overline{I_{FG}I_{GH}G}$ and $\overline{I_{EF}I_{HE}E}$ are isosceles. By the isosceles angle theorem, $\alpha = \rho - \frac{1}{2}\angle G$ and $\beta = \rho - \frac{1}{2}\angle E$; that is, $\angle G = 2(\rho - \alpha)$ and $\angle E = 2(\rho - \beta)$. Thus, $\angle G + \angle E = 2\sigma - 2(\alpha + \beta)$. But $\angle P = \rho$, so, by the angle sum theorem, $\alpha + \beta = \rho$, and $\angle G + \angle E = \sigma$. By the cyclic quadrilateral theorem converse, $\overline{EFGH}$ is cyclic. ■

The following construction suggests that the bi-medials of a tangential quadrilateral and its contact quadrilateral coincide. Verily, but Brianchon's theorem is the work of blue belts.

Construction 5.1 *Given a circle, construct (1) a bi-centric quadrilateral that it is incircle to; and (2) the quadrilateral to which the bi-centric quadrilateral is the pedal quadrilateral of its bi-medial.*

Solution

1. Choose any interior point and draw perpendicular lines through it at any angle to the incenter. Draw radii from the incenter to where these lines intersect the circle, then draw perpendiculars to the radii through these intersection points. Cut off the tails.
2. From the circumcenter of the given quadrilateral, draw radii to the vertices, then draw perpendiculars to the radii through these intersection points. Cut off the tails. ■

Lemma 5.2

Given $\overline{EFG}$ and P long of $\angle G$, let P_E, P_F, P_G be the pedal vertices of P. Then,

1. *$\angle G$ and $\angle P_F P P_E$ are supplementary.*
2. *$\angle EPP_F = \angle EP_GP_F$ and $\angle FPP_E = \angle FP_GP_E$.*

Proof

1. $\angle G$ and $\angle P_F P P_E$ are supplementary by the right cyclic theorem.
2. $\angle EPP_F = \angle EP_GP_F$ by the inscribed angle theorem in the circle that $\overline{EP}$ is a diameter to. Analogously, $\angle FPP_E = \angle FP_GP_E$ in the circle that $\overline{FP}$ is a diameter to. ■

Wallace Theorem I

A point is on the circumcircle of a triangle if and only if the feet of the perpendiculars dropped from it onto the sides or their extensions are collinear.

The line through P_E, P_F, P_G (if it is a line) is called the **Wallace line** of P relative to $\overline{EFG}$. William Wallace is due the credit for this theorem; for a long time, it was misattributed to Robert Simson.

Proof

Use the lemma 5.2 figure, but with P on the circumcircle of $\overline{EFG}$. $\angle G + \angle EPF = \sigma$ by the cyclic quadrilateral theorem and $\angle G + \angle P_F P P_E = \sigma$ by lemma 5.2.1; by transitivity, $\angle EPF = \angle P_F P P_E$. Of $\angle EPP_E$ and $\angle FPP_F$, one must be inside the other; either way, by subtracting the inside angle from equal angles, $\angle EPP_F = \angle FPP_E$. $\angle EP_G P_F = \angle FP_G P_E$, by lemma 5.2.2 and transitivity. By the vertical angles theorem, P_E, P_F, P_G are collinear.

Use the lemma 5.2 figure, but with P_E, P_F, P_G collinear. $\angle EP_G P_F = \angle FP_G P_E$, by the vertical angles theorem, and $\angle EPP_F = \angle FPP_E$, by lemma 5.2.2 and transitivity. Adding or subtracting equal angles to $\angle FPP_F$ or $\angle EPP_E$ yields $\angle EPF = \angle P_F P P_E$. By lemma 5.2.1, $\angle G + \angle P_F P P_E = \sigma$, so $\angle G + \angle EPF = \sigma$. Thus, E, P, F, G are concyclic by the cyclic quadrilateral theorem converse; that is, P is on the circumcircle of $\overline{EFG}$. ■

P_E, P_F, P_G are vertices of the pedal triangle. In textbooks that consider degenerate triangles to be triangles, the Wallace line is a special case of the pedal triangle; it is the degenerate case.

2010 USAMO Problem

Let $\overline{EFGPH}$ be a pentagon inscribed in a semicircle with diameter $\overline{EF}$. The feet of perpendiculars dropped on $\overleftrightarrow{EH}$ and $\overleftrightarrow{FH}$ from P define a line, and the feet of perpendiculars dropped on $\overleftrightarrow{EG}$ and $\overleftrightarrow{FG}$ from P define a line. Prove that these lines make an angle half that of $\angle GOH$ with $O \equiv M_{EF}$.

Solution

The feet P_E, P_F, P_G define the Wallace line of P relative to $\overline{EFG}$. The feet of perpendiculars dropped on $\overleftrightarrow{EH}$ and $\overleftrightarrow{FH}$ from P define the Wallace line of P relative to $\overline{EFH}$. We will call these feet P'_E and P'_F, respectively, to distinguish them from the feet associated with $\overline{EFG}$. Since $\overline{EF}$ is common to both triangles, their Wallace lines cross at $P_G \equiv P'_H$. $\angle GOH$ is the angle subtended at the center by $\overline{GH}$, so it is double any inscribed angle subtended by $\overline{GH}$, such as $\angle GEH$. Let ω be the circle with diameter $\overline{EP}$. By Thales' diameter theorem, P_G, P_F and P'_E all lie on this circle. Within this circle, $\angle P_F E P'_E = \angle P_F P_G P'_E$ by the inscribed angle theorem. But $\angle P_F E P'_E \equiv \angle GEH$, so $\angle P_F P_G P'_E = \frac{1}{2}\angle GOH$. ■

Orange Belt Exit Exam Problem #6

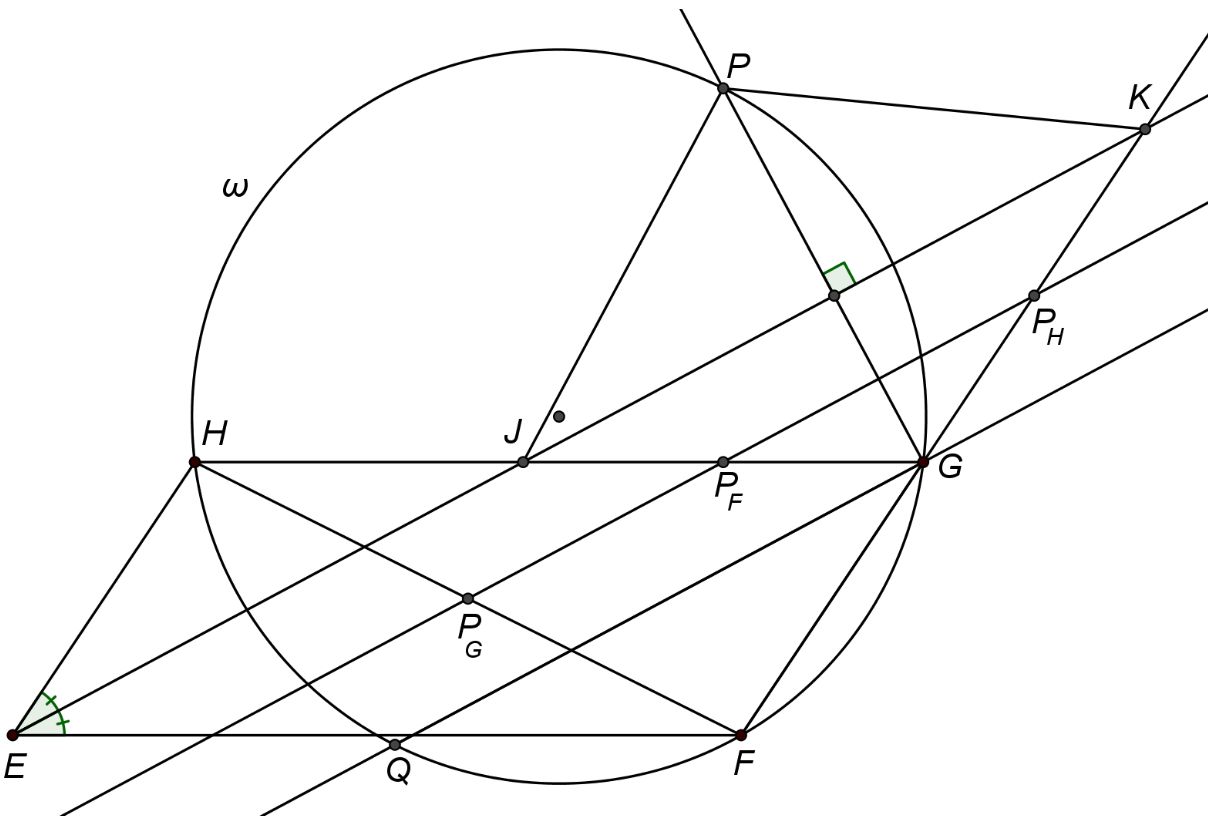
Given a parallelogram $\overline{EFGH}$ that is not a rhombus, draw a ray from E through $\overline{GH}$ at J and through $\overrightarrow{FG}$ at K. Prove that (1) $\overline{EFK} \sim \overline{JHE} \sim \overline{JGK}$; and (2) $\angle E$ is bisected iff $\overline{JG} = \overline{KG}$.

Did everybody get this problem when you were orange belts? You are going to need it!

Isosceles Kite Problem

Photocopy the image in the figure below. Note that $\overline{EFGH}$ is a parallelogram; $\angle E$ is bisected; P is the point on the circumcircle of $\overline{FGH}$, ω, such that $\overline{PG} \perp \overline{JK}$; Q is diametrically opposed to P in ω; and $\overleftrightarrow{P_F P_H}$ is the Wallace line of P relative to $\overline{FGH}$. Prove the following:

1. *$\overline{JGKP}$ is an isosceles kite.*
2. *$\overleftrightarrow{QG} \parallel \overleftrightarrow{JK} \parallel \overleftrightarrow{P_F P_H}$*



Isosceles Kite Problem

Wallace Theorem II

Given $\overline{EFG}$, P on the circumcircle, ω, long of $\angle G$, let $Q := \overleftrightarrow{PP_E} \cap \omega$. Then, $\overleftrightarrow{EQ} \parallel \overleftrightarrow{P_G P_E}$.

Proof

If $Q = \overrightarrow{PP_E} \cap \omega$, $\angle PFE = \angle PQE$ by the inscribed angle theorem. $\angle PP_G F = \angle PP_E F = \rho$, so, by Thales' diameter theorem, P_G and P_E are on the circle whose diameter is $\overline{PF}$. By the inscribed angle theorem, $\angle PFP_G = \angle PP_E P_G$. By transitivity, $\angle PQE = \angle PP_E P_G$. By the vertical angles theorem and the transversal lemma, $\overleftrightarrow{EQ} \parallel \overleftrightarrow{P_G P_E}$. •

If $Q = \overrightarrow{P_E P} \cap \omega$, show that $\angle PFE = \sigma - \angle PQE$; the rest is left as an exercise. • ■

Wallace Theorem III

Given $\overline{EFG}$ and P on the circumcircle, ω, long of $\angle G$; then, $\overline{PEF} \sim \overline{PP_F P_E}$.

Proof

$\angle PP_G F = \angle PP_E F = \rho$, so, by Thales' diameter theorem, P_G and P_E are on the circle whose diameter is $\overline{PF}$. By the inscribed angle theorem, $\angle PFP_G = \angle PP_E P_G$. Analogously, $\angle PEP_G = \angle PP_F P_G$. By AA similarity, $\overline{PEF} \sim \overline{PP_F P_E}$. ■

In the preceding two theorems, the only restriction on P is that it be long of $\angle G$. In the next theorem, P is not just long of $\angle G$, but it is the orthic reflection around $\overline{EF}$. $\overline{EFG}$ is acute to assure that $\angle E$ and $\angle F$ are acute; P is not long of $\angle G$ if $\angle E$ or $\angle F$ are obtuse. $\angle G$ need not be acute.

Wallace Theorem IV

Given $\overline{EFG}$ acute with circumcircle ω, let $P := \overrightarrow{GG'} \cap \omega$, so P_G is G', the foot of the altitude to $\overline{EF}$. The Wallace line determined by P is parallel to the line tangent to ω at G.

Proof

Assume that P_E is outside ω. $\angle FPP_G = \angle FEG$ by the inscribed angle theorem. $\angle PP_G F$ and $\angle PP_E F$ are both right, so, by Thales' diameter theorem, P_G and P_E are on the circle whose diameter is $\overline{PF}$. By the inscribed angle theorem, $\angle FPP_G = \angle FP_E P_G$. By transitivity, $\angle FEG = \angle FP_E P_G$. Let $\overleftrightarrow{GT}$ be tangent to ω at G and T be on the same side of $\overleftrightarrow{GP}$ as F. By the tangent and chord theorem, $\angle FEG = \angle FGT$. By transitivity, $\angle FP_E P_G = \angle FGT$. By the transversal lemma, $\overleftrightarrow{P_G P_E} \parallel \overleftrightarrow{GT}$. •

Case two is that P_E is inside ω and case three is that P_E is on ω; these cases are left as exercises.

Wallace Theorem V

Given $\overline{EFG}$ and P, Q on the circumcircle, ω, both long of $\angle G$, the angle between the Wallace lines determined by P and Q is equal to the angle subtended by $\overline{PQ}$.

Proof

Let $P'' := \overleftrightarrow{PP_E} \cap \omega$ and $Q'' := \overleftrightarrow{QQ_E} \cap \omega$. $\overleftrightarrow{EP''} \parallel \overleftrightarrow{P_G P_E}$ and $\overleftrightarrow{EQ''} \parallel \overleftrightarrow{Q_G Q_E}$ by Wallace theorem II. By the pairwise parallels theorem, $\angle P''EQ''$, the angle subtended by $\overline{P''Q''}$, is equal to the angle between the Wallace lines. By the transversal lemma, $\overleftrightarrow{PP''} \parallel \overleftrightarrow{QQ''}$. By the parallels and circle theorem, and possibly also the isosceles triangle frustum theorem (diagonals are equal), $\overline{PQ} = \overline{P''Q''}$. Thus, the result. ■

Lemma 5.3

An interior angle of one equilateral triangle is equal to an interior angle of any equilateral triangle.

> ***Proof***
>
> By the equilateral triangle theorem, all three angles are equal. Choose any side as the base and, by SSS, construct a triangle congruent to the other triangle with collinear bases and sharing a base vertex. By C. 3.3, construct a line through its apex parallel to the base. By T & V, the interior angles of this triangle are equal to the given triangle. ■

Lobachevski postulated that, given a line and a point not on it, there are at least two lines through that point parallel to the given line. This means that: (1) There are no right rectangles – a Lambert quadrilateral's fourth angle is acute; (2) The sum of a triangle's interior angles is less than straight; and (3) No two triangles are similar unless they are also congruent. Thus, in hyperbolic geometry, equilateral triangles are equiangular, but that angle depends on the size of the triangle.

Thus, Torricelli's problem (next) can be solved in Euclidean space. But, if the vertices were Earth and two exoplanets, would we know where to construct a space port between them? Maybe.

It is possible that physical space is Euclidean on a small scale but becomes hyperbolic as distances become greater. In hyperbolic geometry, the sum of the angles in a triangle decreases as size increases – which is why there are no similar triangles except those that are also congruent – but for very small triangles the angle sum is so close to straight that we may not be able to measure its defect with our instruments. But what does "small" mean in the real world? The width of my desk? The width of Germany? The width of the solar system? Thanks to Carl Friedrich Gauss, we know that the triangle with vertices on the peaks of the mountains Hohenhagen, Inselberg and Brocken is small in this sense. We wait for NASA to inform us if the triangle with vertices on Earth, Jupiter and Saturn is small. Until this measurement is taken, we really do not know if the universe is Euclidean all the way out or just locally where we have line-of-sight with telescopes.

Recall the notation established in lemma 3.6 that φ (phi) is the interior angle of an equilateral triangle. It is a third of a straight angle in Euclidean geometry; in hyperbolic geometry, it is less.

The next problem was posed by Pierre de Fermat, who is famous for his Last Theorem. Decades later it was solved by Evangelista Torricelli, who is famous for inventing the barometer, and then solved using a different method by Napoleon Bonaparte, who is famous for conquering Europe.

For not **too obtuse** triangles – the interior angles are all less than 2φ – **Torricelli segments** are from the vertices to the apexes of equilateral triangles built on the exterior of the opposite sides.

Torricelli Lemma: *The Torricelli segments are concurrent; this point is called the* ***Torricelli point****.*

Torricelli Problem: *Given a triangle that is not too obtuse (interior angles all less than 2φ), prove that the Torricelli point minimizes the sum of the distances to the triangle's vertices.*

> ***Solution***
>
> Given a triangle $\overline{EFG}$, guess where the desired center, U, is. Build an equilateral triangle on $\overline{EU}$ with its apex, P, on the other side of $\overleftrightarrow{EU}$ from F so $\overline{EP} = \overline{UP}$. Build an equilateral triangle on $\overline{EG}$ so its apex, F'', is outside the given triangle. $\angle UEG = \angle PEF''$ because they are both $\angle GEP$ less than (more than) φ if P is outside (inside) $\overline{EFG}$. $\overline{UEG} \cong \overline{PEF''}$ by SAS and so $\overline{UG} = \overline{PF''}$; thus, $\overline{FU} + \overline{EU} + \overline{GU} = \overline{FU} + \overline{EP} + \overline{PF''} = \overline{FU} + \overline{UP} + \overline{PF''}$. This distance is shortest if F, U, P, F'' are collinear; that is, U is on the line from a vertex of the given triangle to the apex of an equilateral triangle built on the exterior of the opposite side. Do the same with another vertex and its opposite side; where the lines cross is the point that minimizes the sum of the distances to the triangle's vertices. ■

For too obtuse triangles, it is easy to show that the solution is at the wide vertex. In this case, the solution is not called the Torricelli point unless that interior angle is exactly 2φ.

Torricelli Angles Theorem

U is the Torricelli point of $\overline{EFG}$ if and only if $\angle EUF = \angle FUG = \angle GUE = 2\varphi$.

This is an easy corollary that uses the same figure as the Torricelli problem. Since $\overline{EUP}$ was constructed equilateral, $\angle EUF$ is exterior to it. Analogously, for $\angle FUP$ and $\angle GUP$.

Define the **Torricelli apexes** E'' and G'' analogous to F'', above. What can we say about $\overline{E''F''G''}$?

Torricelli Expansion Theorem

$\overline{EFG}$ and $\overline{E''F''G''}$, with E'', F'', G'' the Torricelli apexes of $\overline{EFG}$, have the same Torricelli point.

Torricelli Segments Theorem

The Torricelli segments are of equal length.

> ***Proof***
>
> Given $\overline{EFG}$ and the Torricelli apexes, E'' and G'', connect $\overline{EE''}$ and $\overline{GG''}$. $\overline{EF} = \overline{G''F}$ and $\overline{FE''} = \overline{FG}$ by equilateralism and $\angle EFE'' = \angle G''FG$ because they are both $\angle EFG + \varphi$. By SAS, $\overline{EFE''} \cong \overline{G''FG}$ and thus $\overline{E''E} = \overline{GG''}$. ■

Torricelli Triangles Circumcircles Theorem

The circumcircles of the three external equilateral triangles are concurrent at the Torricelli point.

Proof

Consider the quadrilateral with three of its corners at the vertices of one of the external equilateral triangles and the fourth at the Torricelli point. By the Torricelli angles theorem, the interior angle at this point and the angle opposite it are supplementary; by the cyclic quadrilateral theorem converse, this quadrilateral is cyclic. Since circumcircles are unique, the triangle's and the quadrilateral's circumcircles are identical. ■

Tri–Segment Theorem

A **tri-segment** *of a triangle is parallel to the other side and a third of it, if it is the one close to the apex; or two-thirds of it, if it is the one close to the base.*

Tri–Segment Theorem Converse

Two lines parallel to the base of a triangle that trisect one side also trisect the other side.

Napoleon Theorem

Centers of equilateral triangles external to a not too obtuse triangle are an equilateral triangle.

Medial Point Proof

Given $\overline{EFG}$ with medial point C, let G'' be the apex of the equilateral triangle built on the exterior of $\overline{EF}$ and C_G its medial point. Let M_{EF} be the midpoint of $\overline{EF}$ and draw the medians $\overline{M_{EF}G}$ and $\overline{M_{EF}G''}$; also, connect $\overline{CC_G}$. By the two-to-one medial point theorem, $\overline{M_{EF}C_G}$ is a third of $\overline{M_{EF}G''}$ and $\overline{M_{EF}C}$ is a third of $\overline{M_{EF}G}$. By the SAS third-scale triangle theorem, $\overline{CC_G}$ is a third of $\overline{GG''}$. By analogy, $\overline{CC_E}$ is a third of $\overline{EE''}$ and $\overline{CC_F}$ is a third of $\overline{FF''}$, where C_E and C_F are medial points of $\overline{GFE''}$ and $\overline{EGF''}$, respectively. By the tri-segment theorem, $\overleftrightarrow{CC_E} \parallel \overleftrightarrow{EE''}$ and $\overleftrightarrow{CC_F} \parallel \overleftrightarrow{FF''}$ and $\overleftrightarrow{CC_G} \parallel \overleftrightarrow{GG''}$. By the Torricelli angles theorem, the angles around the Torricelli point are 2φ and, by the pairwise parallel similarity theorem, the angles around C are 2φ; thus, $\overline{C_E C_F C_G}$ is equilateral. ■

The center of this equilateral triangle is the **first Napoleon point**. This theorem is also true of the three internal equilateral triangles; the center of that equilateral triangle is the **second Napoleon point**. Donald Coxeter's circumcenter proof is green belt; it extends this to too obtuse triangles.

Moss Problem

Construct the largest equilateral triangle, $\overline{EFG}$, with given points J, K, L, each on a different side.

Lemma 5.4

Given $\overline{EFG}$ equilateral with center O, then $\angle EOF = \angle FOG = \angle EOG = 2\varphi$

Equilateral Sum Theorem

Given $\overline{EFG}$ equilateral and P on its circumcircle long of $\angle G$, then $\overline{GP} = \overline{FP} + \overline{EP}$.

Proof

Assume that P is closer to E than to F; if not, then change the labels. Drop perpendiculars from the circumcenter, O, to $\overline{EP}, \overline{FP}, \overline{GP}$ with feet N, J, M. By the diameter and chord theorem, these are midpoints. Drop a perpendicular from E to $\overline{OJ}$ with foot K and then from P to $\overline{EK}$ with foot L. By the Lambert theorem, $\overline{PJKL}$ is a right rectangle; $2\overline{KL} = \overline{FP}$. By the cyclic quadrilateral theorem, $\angle EPF = 2\varphi$. By lemma 3.6, $\overline{EPL}$ is a half equilateral triangle, so $2\overline{LE} = \overline{PE}$. Thus, $2\overline{KE} = 2\overline{KL} + 2\overline{LE} = \overline{FP} + \overline{EP}$ by substitution.

If $\overline{MP} = \overline{KE}$, then $\overline{GP} = 2\overline{KE}$ and we are done!

$\overline{ONPJ}$ is cyclic (opposite angles right) and we know that $\angle NPJ = 2\varphi$; thus, $\angle NOJ = \varphi$.

$\angle MOP$	$= \frac{1}{2}\angle POG$	center line theorem
	$= \frac{1}{2}(\angle POE + \angle EOG)$	addition
	$= \frac{1}{2}(2\angle NOE + 2\varphi)$	center line theorem and lemma 5.4
	$= \angle NOE + \varphi$	cancel half of double
	$= \angle NOE + \angle NOJ$	substitution
	$= \angle JOE = \angle KOE$	addition and collinearity

By AAS, $\overline{MOP} \cong \overline{KOE}$, which holds the equality $\overline{MP} = \overline{KE}$. ■

Reverse Torricelli Problem

Given that E'', F'', G'' are the Torricelli apexes of $\overline{EFG}$, construct $\overline{EFG}$.

Solution

By the Torricelli expansion theorem, $\overline{EFG}$ and $\overline{E''F''G''}$ have the same Torricelli point; find it and call it U. By the equilateral sum theorem, $\overline{UE} + \overline{UF} = \overline{UG''}$ and $\overline{UF} + \overline{UG} = \overline{UE''}$ and $\overline{UG} + \overline{UE} = \overline{UF''}$. Add them together: $2\left(\overline{UE} + \overline{UF} + \overline{UG}\right) = \overline{UE''} + \overline{UF''} + \overline{UG''}$. By substitution, $\overline{UE} = \frac{1}{2}\left(\overline{UF''} + \overline{UG''} - \overline{UE''}\right)$ and $\overline{UF} = \frac{1}{2}\left(\overline{UE''} + \overline{UG''} - \overline{UF''}\right)$ and $\overline{UG} = \frac{1}{2}\left(\overline{UE''} + \overline{UF''} - \overline{UG''}\right)$. Lay these lengths off on $\overrightarrow{E''U}$ and $\overrightarrow{F''U}$ and $\overrightarrow{G''U}$ past U, respectively. These are the vertices of $\overline{EFG}$, respectively. ■

Lemma 5.5
Of isosceles triangles with equal apex angles, the one with the shortest legs has the shortest base.

Fagnano Problem: *Inscribe a triangle in an acute triangle with the smallest possible perimeter.*

Solution
Given $\overline{EFG}$, guess at the inscribed triangle's vertex on $\overline{EF}$; call this P. Let P_{GE} and P_{FG} be reflections of P around $\overleftrightarrow{GE}$ and $\overleftrightarrow{FG}$, respectively. Let $F'' := \overline{P_{GE}P_{FG}} \cap \overline{GE}$ and $E'' := \overline{P_{GE}P_{FG}} \cap \overline{FG}$. The perimeter of $\overline{PE''F''}$ equals $\overline{P_{GE}P_{FG}}$ by the mediator theorem and it is the minimal inscribed triangle with P on $\overline{EF}$ by definition of segment. But which point P on $\overline{EF}$ has the shortest segment $\overline{P_{GE}P_{FG}}$ associated with it?

For any P on $\overline{EF}$; by SSS, $\overline{PGF''} \cong \overline{P_{GE}GF''}$ and $\overline{PGE''} \cong \overline{P_{FG}GE''}$, so $\angle PGF'' = \angle P_{GE}GF''$ and $\angle PGE'' = \angle P_{FG}GE''$. $\angle P_{GE}GP_{FG} = 2\angle EGF$, a given angle. $\overline{P_{GE}G} = \overline{P_{FG}G} = \overline{PG}$, so $\overline{P_{GE}P_{FG}G}$ is isosceles with a given apex angle. By the perpendicular length theorem, $\overline{PG}$ is minimal if it is the perpendicular from G to $\overline{EF}$; that is, $P \equiv G'$. By lemma 5.5, this minimizes $\overline{P_{GE}P_{FG}}$. If $E'' \equiv E'$ and $F'' \equiv F'$, then the solution is the orthic triangle.

Let $\angle E = \alpha, \angle F = \beta, \angle G = \gamma$ and α', β', γ' be their complements. $\angle G'_{GE}GG'_{FG} = 2\gamma$ and, by the isosceles angle theorem, $\angle GG'_{GE}F'' = \gamma'$. $\angle G'GE = \alpha'$, so $\angle G'_{GE}GF'' = \alpha'$. By the exterior angle theorem, $\angle G'F''E = \alpha' + \gamma' = \beta$. $\angle G'F''G = \sigma - \beta$, so, by the cyclic quadrilateral theorem converse, $\overline{G'FGF''}$ is cyclic. By Thales' diameter theorem, $\overline{FG}$ is a diameter, $\angle FF''G = \rho$ and $F'' \equiv F'$. Analogously, $E'' \equiv E'$. It's the orthic triangle! ■

The first two paragraphs cite only yellow-belt theorems and, had we stopped there and said, "by analogy, we could have done this for P on $\overline{FG}$ or P on $\overline{GE}$ and found E' and F', respectively," as other authors do, it would imply that this works in hyperbolic geometry. *But it does not!*

Orthic Triangle Lemma
Two triangle vertices and the feet of the altitudes from them are concyclic.

Proof
By Thales' diameter theorem, E, F, E', F' are concyclic and centered at the midpoint of $\overline{EF}$. By analogy, F, G, F', G' and G, E, G', E' are concyclic. ■

Orthic Triangle Similarity Theorem
The orthic triangle of an acute triangle cuts off three triangles from it that are similar to it.

Proof

By the orthic triangle lemma, E, F, E', F' are concyclic. By the cyclic quadrilateral theorem, $\angle F'EF$ and $\angle FE'F'$ are supplementary. $\angle F'E'G$ and $\angle FE'F'$ are supplementary by construction, so $\angle F'EF = \angle F'E'G$. Also, $\angle E'FE = \angle E'F'G$. By AA similarity, $\overline{EFG} \sim \overline{E'F'G}$. Analogously, $\overline{EFG} \sim \overline{EF'G'} \sim \overline{E'FG'} \sim \overline{E'F'G}$. ■

Orthic Circumradius Theorem

The circumradii to a vertex and a side of the orthic triangle are perpendicular.

Proof

Given $\overline{EFG}$ with circumcenter O, orthocenter H and orthic reflections H_{FG}, H_{GE}, H_{EF}, let $M := \overrightarrow{GO} \cap \overline{H_{FG}H_{GE}}$. By the orthic triangle lemma, E, F, E', F' are concyclic. By the inscribed angle theorem, $\angle E'EF' = \angle E'FF'$. Their complements, $\angle EGE' = \angle FHE'$, are equal. $\angle EGF = \angle EH_{FG}F$ by the inscribed angle theorem. $\angle FHE' = \angle FH_{FG}E'$ by transitivity. By AAS, $\overline{FHE'} \cong \overline{FH_{FG}E'}$; thus, $\overline{HE'} = \overline{H_{FG}E'}$. Analogously, $\overline{HF'} = \overline{H_{GE}F'}$. By mid-segment theorem #1, $\overleftrightarrow{E'F'} \parallel \overleftrightarrow{H_{FG}H_{GE}}$.

From above, $\angle E'EF' = \angle E'FF'$. The same equality is $\angle H_{FG}EG = \angle GFH_{GE}$. By the inscribed angle theorem, $\angle H_{FG}OG = \angle GOH_{GE}$. By the center line theorem, $\angle M$ is right. By the transversal theorem corollary, the result. ■

We call the medial point C and never use it nor A, B and D for arbitrary points. It is weak that high-school math teachers overuse the letters at the beginning and end of the alphabet, which confuses students as they move from one math class to another and keep seeing the same letters used with different meanings. The Pythagorean theorem and the quadratic formula are to tradesmen what the one-two punch is to pugilists. Yet most cannot solve these common math problems, largely because their geometry teacher used a, b, c for the sides of right triangles and their algebra teacher used a, b, c for the coefficients of quadratic equations, $ax^2 + bx + c = 0$.[95] Labeling the circumcenter C is also confusing because that is a vertex. It will soon be proven that the circumcenter of a triangle is the orthocenter of its medial triangle, so it is impossible for them to have unique symbols, but I try to use O for circumcenter and H for orthocenter, if possible. Triangle vertices E, F, G, medial point C, and incenter I are the only unique unaccented symbols. A single prime always means the foot of an altitude; *e.g.,* E', F', G'. Double subscripts on I are touching points of a quadrilateral; on M they are midpoints. A parent and medial triangle have equal angles and should be written in that order; *e.g.,* $\overline{M_{FG}M_{GE}M_{EF}}$ is medial to $\overline{EFG}$. Otherwise, double subscripts are reflections; thus, H_{FG}, H_{GE}, H_{EF} are the orthic reflections of $\overline{EFG}$.

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[95] Here I help tradesmen: www.researchgate.net/publication/282947903_How_Math_Can_Be_Taught_Better

Orthic Triangle Incenter Theorem

1. *The orthocenter of an acute triangle is the incenter of its orthic triangle.*
2. *The obtuse vertex of an obtuse triangle is the incenter of its orthic triangle.*

Proof of First Case

Given $\overline{EFG}$ with orthocenter H, by the cyclic quadrilateral theorem converse, $\overline{EG'HF'}$ is cyclic. By the inscribed angle theorem, $\angle F'EH = \angle F'G'H$ and $\angle E'FH = \angle E'G'H$. By the inscribed angle theorem converse, $\overline{EFE'F'}$ is cyclic and, by the inscribed angle theorem, $\angle F'EH = \angle E'FH$. By the preceding equalities, $\angle F'G'E'$ is bisected. Analogously, $\angle G'E'F'$ and $\angle E'F'G'$ are bisected. Thus, H is the incenter of $\overline{E'F'G'}$. •

We will now prove a result due to Leonhard Euler, who is famous in both geometry and calculus; his $e^{i\theta} = \cos\theta + i\sin\theta$ founded complex analysis. Also, he is the founder of modern ballistics.

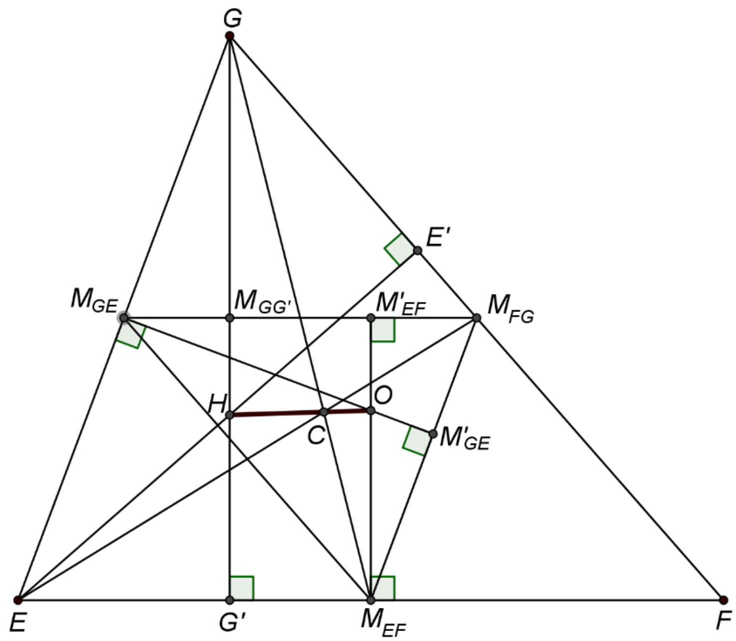
Euler Segment Theorem

The medial point is collinear with the orthocenter and the circumcenter and twice as far from the former as the latter.

Proof

Given $\overline{EFG}$, construct the medial triangle $\overline{M_{FG}M_{GE}M_{EF}}$. By medial triangle theorem I, $\overline{M_{FG}M_{GE}} = \frac{1}{2}\overline{EF}$ and $\overline{M_{GE}M_{EF}} = \frac{1}{2}\overline{FG}$ and $\overline{M_{EF}M_{FG}} = \frac{1}{2}\overline{GE}$. Raise a perpendicular from $\overline{EF}$ at M_{EF} that intersects $\overline{M_{FG}M_{GE}}$ at M'_{EF}. Raise a perpendicular from $\overline{EG}$ at M_{GE} that intersects $\overline{M_{EF}M'_{EF}}$ at O. By the medial triangle orthocenter theorem, these segments are altitudes of the medial triangle and thus their intersection, O, is both the orthocenter of the medial triangle and the circumcenter of the parent triangle. Construct altitudes from E and G, the latter intersecting $\overline{M_{FG}M_{GE}}$ at $M_{GG'}$ and $\overline{EF}$ at G'. By the orthocenter theorem, the intersection of these altitudes, H, is the orthocenter of $\overline{EFG}$. $\overline{HO}$ is the **Euler segment** of $\overline{EFG}$. By the two-to-one medial point theorem, $C := \overline{EM_{FG}} \cap \overline{GM_{EF}}$ is the medial point and $\overline{M_{EF}C} = \frac{1}{2}\overline{GC}$. By the half-scale orthocenter to vertex theorem, $\overline{OM_{EF}} = \frac{1}{2}\overline{HG}$. By the Lambert theorem, $\overline{M_{EF}M'_{EF}M_{GG'}G'}$ is a right rectangle. Thus, $\overleftrightarrow{GG'} \parallel \overleftrightarrow{M_{EF}M'_{EF}}$ and, by the transversal theorem, $\angle OM_{EF}G = \angle HGM_{EF}$. Thus, $\overline{HGC}$ has two sides that are half the corresponding sides in $\overline{OM_{EF}C}$ and the included angle is equal. By the SAS half-scale triangle theorem, $\overline{OC} = \frac{1}{2}\overline{HC}$ and $\angle OCM_{EF} = \angle HCG$. By the vertical angles theorem, C is collinear with H and O. ■

The Euler line is the segment extended; neither exist for equilateral triangles because $H \equiv O$.



By the center line theorem, the incenter is collinear with the Euler segment if and only if the triangle is isosceles. But is it between the orthocenter and the circumcenter and thus on the Euler segment? The Euler segment relates three major triangle centers but locating the incenter relative to it has long stymied geometers. Guinand (1984) and Franzsen (2011) made progress!

Guinand's Theorem (without proof)

For any non-equilateral triangle, the incenter lies strictly inside and the excenters lie strictly outside the circle whose diameter joins the medial point to the orthocenter.

In 2001, Várilly proved that the Torricelli point is in the same circle that Guinand contained the incenter. Then, in 2011, Franzsen proved the following theorem.

Franzsen's Theorem (without proof)

Let d be the distance from the incenter to the Euler segment, s the semiperimeter, μ (Greek: mu) the longest side and ν (Greek: nu) the longest median. Then, $\frac{d}{s} < \frac{d}{\mu} < \frac{d}{\nu} < \frac{1}{3}$.

Proofs are beyond the scope of this book, but this paper[96] leads to questions that occupy modern geometers and are of interest to top students who want to participate alongside the professors. It is absurd that Agostino Prástaro[97] denounces us for believing that mathematics ends with Euclid. I never said that it did. Indeed, the brilliant teenager is more likely to get published in a refereed journal if he pursues geometry than if he takes up any other scientific inquiry.

[96] forumgeom.fau.edu/FG2011volume11/FG201126.pdf

[97] davidlowryduda.com/reviewing-goldbach/#more-706

Orthocenter and Wallace Line Theorem

The Wallace line determined by P and $\overline{EFG}$ bisects $\overline{PH}$, with H the orthocenter.

$\angle E$ can be acute, right or obtuse; in each case, $\overrightarrow{PP_G}$ cuts ω, or $\overrightarrow{P_GP}$ cuts ω, or $\overleftrightarrow{PP_G}$ touches ω. Here, $\angle E$ acute and $\overrightarrow{PP_G}$ cuts ω. I will shake the hand of the man who proves all nine cases!

Proof

Given $\overline{EFG}$ with P on the circumcircle, ω, between E and G, let $Q := \overrightarrow{PP_G} \cap \omega$ and $M := \overline{EF} \cap \overline{PH_{EF}}$ and $P'' := \overrightarrow{PQ} \cap \overrightarrow{HM}$. By the transversal lemma, $\overleftrightarrow{PP''} \parallel \overleftrightarrow{GH_{EF}}$. By the orthocenter and circumcircle theorem, $\overline{G'H} = \overline{G'H_{EF}}$ and, $\overline{MG'H} \cong \overline{MG'H_{EF}}$ by SAS, so $\overline{MH} = \overline{MH_{EF}}$. $\angle PQG = \angle PH_{EF}G = \angle P''HH_{EF} = \angle PP''H = \angle H_{EF}PP''$ by the inscribed angle, isosceles triangle, transversal and transversal theorems. By T & V and Wallace theorem II, $\overleftrightarrow{P''H} \parallel \overleftrightarrow{QG} \parallel \overleftrightarrow{P_GP_E}$. By AAS, $\overline{PP_GM} \cong \overline{P''P_GM}$, so $\overline{PP_G} = \overline{P''P_G}$; that is, P_G is the midpoint of $\overline{P''P}$, so P'' can be renamed P_{EF}. By mid-segment theorem #2, the Wallace line determined by P bisects $\overline{PH}$. •

Euler Circle Lemma

Given $\overline{EFG}$ with E', F', G' the feet of the altitudes, then $E', F', G', M_{EF}, M_{FG}, M_{GE}$ are concyclic.

Proof

Given $\overline{EFG}$, assume that G' is between E and M_{EF}; if it is not, re-label E and F. By mid-segment theorem #1, $\overline{M_{EF}M_{FG}} = \frac{1}{2}\overline{EG}$. By Thales' diameter theorem, $\overline{M_{GE}G'} = \frac{1}{2}\overline{EG}$. By transitivity, $\overline{M_{EF}M_{FG}} = \overline{M_{GE}G'}$, so $\overline{M_{GE}G'M_{EF}M_{FG}}$ is an isosceles triangle frustum. By mid-segment theorem #1, $\overleftrightarrow{G'M_{EF}} \parallel \overleftrightarrow{M_{GE}M_{FG}}$. By the isosceles triangle frustum theorem, $G', M_{EF}, M_{FG}, M_{GE}$ are concyclic. Analogously, E' and F' are on this circle. ■

We tacitly used the circumcenter theorem corollary; any three noncollinear points fully define a circle. One can also say that the orthic and the medial triangles have the same circumcircle.

Euler Circle Theorem

Given $\overline{EFG}$ with E', F', G' the feet of the altitudes and H the orthocenter, the following nine points are concyclic: E', F', G' and M_{EF}, M_{FG}, M_{GE} and M_{EH}, M_{FH}, M_{GH}.

Proof

Consider $\overline{EFH}$. The feet of its altitudes are E', F', G' and, by the Euler circle lemma, they are concyclic with M_{EH} and M_{FH}. Analogously, M_{GH} is on this circle. E', F', G' fully define a circle so, by the Euler circle lemma, M_{EF}, M_{FG}, M_{GE} are also on this circle. ■

Euler Center Theorem
The center of the ***Euler circle*** *is the midpoint of the Euler segment.*

Proof
Recall the figure for the proof of the Euler segment theorem. $\overline{G'M_{EF}OH}$ is a triangle frustum. By the diameter and chord theorem, the mediator of $\overline{G'M_{EF}}$ is a diameter of the Euler circle because $\overline{G'M_{EF}}$ is a chord. By the triangle frustum mid-segment theorem converse, it bisects the Euler segment, $\overline{OH}$. Analogously, the mediators of $\overline{E'M_{FG}}$ and $\overline{F'M_{GE}}$ bisect $\overline{OH}$. Thus, the result, because diameters intersect at the center. ■

Euler Radius Theorem
The radius of a triangle's Euler circle is half its circumradius.

Proof
Given $\overline{EFG}$, let O, N, H be the circumcenter, **Euler center** and orthocenter. By the SAS half-scale triangle theorem, $\overline{NHM_{GH}}$ is half the lengths of $\overline{OHG}$, so $\overline{NM_{GH}} = \frac{1}{2}\overline{OG}$. ■

Euler Diameter Theorem
$\overline{M_{EH}M_{FH}M_{GH}} \cong \overline{M_{FG}M_{GE}M_{EF}}$ *and* $\overline{M_{EH}M_{FG}}, \overline{M_{FH}M_{GE}}, \overline{M_{GH}M_{EF}}$ *are diameters of the Euler circle.*

Given $\overline{EFG}$, we know that M_{EH}, M_{FH}, M_{GH} are on the Euler circle. But does the Euler circle bisect the segment between the orthocenter and *any* point on the circumcircle?

Euler Bisection Theorem
The Euler circle bisects any segment from the orthocenter to the circumcircle.

Proof
Given $\overline{EFG}$, let O, N, H be the circumcenter, Euler center and orthocenter; O_1 is on the circumcircle. For any N_1 on the Euler circle, $\overline{NN_1} = \frac{1}{2}\overline{OO_1}$ by the Euler radius theorem. Find N_1 such that it is on the same side of $\overleftrightarrow{HO}$ as O_1 and $\overleftrightarrow{NN_1} \parallel \overleftrightarrow{OO_1}$. Connect $\overline{HO_1}$. By the Euler center theorem, $N \equiv M_{HO}$. With $\overline{NN_1} = \frac{1}{2}\overline{OO_1}$, we have all the conditions for mid-segment theorem #3 regarding $\overline{OO_1H}$; thus, $N_1 \equiv M_{HO_1}$. O_1 is any point on the circumcircle and N_1 is a point on the Euler circle that bisects $\overline{HO_1}$. ■

Problem 5.2 *Prove that the circumcircle of a triangle is the Euler circle of a triangle whose vertices are the given triangle's incenter and two of its excenters.*

Problem 5.3 *Given $\overline{EFG}$, prove that O, M_{EF}, F, M_{FG} are concyclic and that this circle is congruent to the Euler circle of $\overline{EFG}$.*

A **homothetic double** of a triangle has sides twice the lengths of the sides of the given triangle and side extensions pairwise parallel. Blue belts will learn that homothetic dilations can be any proportion, but we have not yet defined proportions. Homothecy is introduced here only for doubling of the lengths. Orange belts had two examples, though they did not then call them this:

1. $\overline{EFG}$ is the homothetic double of $\overline{M_{GE}M_{FG}G}$.

2. The parent triangle, $\overline{EFG}$, is the homothetic double of the medial triangle, $\overline{M_{FG}M_{GE}M_{EF}}$.

In example #1, $G = \overleftrightarrow{EM_{GE}} \cap \overleftrightarrow{FM_{FG}}$ because the sides are collinear, so lines through the vertices taken pairwise either intersect at G or the vertices are G. In example #2, by the medial and parent triangle theorem, the medial triangle and its parent triangle have the same medial point, so $C := \overleftrightarrow{EM_{FG}} \cap \overleftrightarrow{FM_{GE}} \cap \overleftrightarrow{GM_{EF}}$. In examples #1 and #2, G and C are the **homothetic centers**.

Double–Long Triangle Theorem I
$\overline{EFG}$ is the orthic triangle of its double-long triangle, $\overline{XYZ}$.

Double–Long Triangle Theorem II
The double-long triangle is a homothetic double of the long triangle.

Proof
By the long circle theorem, L_E, L_F, L_G are the midpoints of $\overline{IZ}, \overline{IY}, \overline{IX}$, respectively. By mid-segment theorem #1, $\overleftrightarrow{XY}, \overleftrightarrow{YZ}, \overleftrightarrow{ZX}$ are pairwise parallel to $\overleftrightarrow{L_FL_G}, \overleftrightarrow{L_EL_F}, \overleftrightarrow{L_GL_E}$, and $\overline{XY}, \overline{YZ}, \overline{ZX}$ are twice as long as $\overline{L_FL_G}, \overline{L_EL_F}, \overline{L_GL_E}$, respectively. The incenter of $\overline{EFG}$ is the homothetic center of $\overline{XYZ}$ and $\overline{L_EL_FL_G}$. ■

Double–Long Triangle Theorem III
The circumcircle of a triangle is the Euler circle of its double-long triangle.

Proof
By the long triangle theorem, the incenter of a triangle is the orthocenter of its long triangle. By the double-long triangle theorem II, the side extensions are parallel; thus, by the transversal theorem, the incenter is also the orthocenter of the double-long triangle. By the long circle theorem, the long centers are midway between the double-long triangle's orthocenter and its vertices. Thus, by the circumcenter theorem corollary. ■

Double–Scale Chords Theorem

Given a circle of radius r, center O_1 and $\overline{TQ_1}$ a chord on it, the locus of points, Q_2, such that $\overline{TQ_2} = 2\overline{TQ_1}$ and T, Q_1, Q_2 are collinear is a circle of radius $2r$ tangent to the given circle at T.

Proof

Let O_2 be the center of the circle of radius $2r$ tangent to the given circle at T. For any Q_1 on the O_1-circle, let Q_2 be the intersection of $\overleftrightarrow{Q_1T}$ and the O_2 circle. By the vertical angles theorem or reflexivity (if the circles are disjoint or not, respectively) and isosceles triangle theorem, $\angle O_1Q_1T = \angle O_1TQ_1 = \angle O_2TQ_2 = \angle TQ_2O_2$. By the AAS half-scale triangle theorem, $\overline{TQ_2O_2}$ is a homothetic double of $\overline{TQ_1O_1}$, so $\overline{TQ_2} = 2\overline{TQ_1}$. ■

Problem 5.4 *Through one of two points of intersection of two circles, draw a line so the circles cut off two chords, one double the length of the other.*

Solution

Let O_1 be the center of one circle and P be a point of intersection. By the double-scale chords theorem, the locus of endpoints of double-length chords is a circle of twice the radius and tangent to the O_1-circle at P. By the common point theorem, its center is collinear with O_1 and P. Where it intersects the other given circle is an endpoint. ■

Discussion

Because the problem did not specify which circle gets the longer chord, there are two solutions. There are always exactly two because the given circles are not tangent. ■

Problem 5.5 *Given an angle $\angle EFG$ and a point P not on either ray of the angle, draw a line through P that intersects $\overrightarrow{FE}$ at J and $\overrightarrow{FG}$ at K so $\overline{PJ}$ is double $\overline{PK}$.*

Two Solutions

1. Assume that P is on the G side of $\overleftrightarrow{FE}$; if not, then re-label. Let L be an arbitrary point on $\overrightarrow{FE}$. By C. 3.3, construct a line through M_{PL} parallel to $\overleftrightarrow{FE}$. K_1 is its intersection with $\overrightarrow{FG}$. $J_1 := \overleftrightarrow{PK_1} \cap \overleftrightarrow{FE}$. By mid-segment theorem #2, $\overline{PJ_1} = 2\overline{PK_1}$.

2. Extend $\overrightarrow{LP}$ by half $\overline{LP}$ to N. By C. 3.3, construct a line through N parallel to $\overleftrightarrow{FE}$. K_2 is its intersection with $\overrightarrow{FG}$. $J_2 := \overleftrightarrow{PK_2} \cap \overleftrightarrow{FE}$. By the transversal, vertical angles and ASA half-scale triangle theorems, $\overline{NPK_2}$ is half the lengths of $\overline{LPJ_2}$; thus, $\overline{PJ_2} = 2\overline{PK_2}$. ■

We cannot yet solve problems like this for arbitrary proportions, but **triple** is as easy as double.

The tri-segment theorem and its converse were needed for the Napoleon theorem to prove that the tri-segment and the base extensions are parallel, but the tri-segment being either one- or two-thirds of the base is also very useful. It is easy to extend this to the quartile points. Blue belts will prove the side-splitter theorem, which extends this to any proportion and is something that every *Common Core* student memorizes. But memorizing facts is what history class is for; we are here to learn logic! Until we prove the extension, we will just use $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

Problem 5.6 *Given an angle $\angle EFG$ and a point P not on either ray of the angle, draw a line through P that intersects $\overrightarrow{FE}$ at J and $\overrightarrow{FG}$ at K so $\overline{PJ}$ is triple $\overline{PK}$.*

Most homothecy problems are solved without knowing or learning what the proportion is. For this to work, we must know that it works for all real numbers, but they were not fully defined until two thousand years after Euclid. We will solve such a problem now, but without proving it.

Problem 5.7 *Inscribe a square inside an equilateral triangle.*

Solution (without proof)
Given $\overline{EFG}$ equilateral, construct $\overline{EFF''E''}$ square and on the same side of $\overleftrightarrow{EF}$ as G. Let $J := \overline{E''M_{EF}} \cap \overline{EG}$ and $K := \overline{F''M_{EF}} \cap \overline{FG}$. J' and K' are the feet of perpendiculars dropped on $\overleftrightarrow{EF}$ from J and K, respectively. $\overline{JKK'J'}$ is square and it is inscribed in $\overline{EFG}$. ■

When I was a 14-year-old geometry student in high school, the teacher inscribed an equilateral triangle, a square and a hexagon in a circle and then gave us problem 5.7 as a challenge problem, even though similarity and homothecy was still in our future. I solved it! Not Eddie Opitz or any other of those guys. But I could not prove $\overline{JKK'J'}$ square. I knew only of mid-segment theorem #1, but $\overline{JKK'J'}$ is not half scale, and I did not have the side-splitter theorem yet. Neither do you.

Homothetic doubles were defined for triangles, but we can also say that the circumcircle is the homothetic double of the Euler circle because its radius is double, and its center is twice as far from the orthocenter. For the Euler circle, this terminology is just another way of stating the Euler bisection theorem. But students should be familiar with this usage because, as blue belts, they will solve problems by constructing circles whose radii are the same proportion as that of the distances of their centers from a point on their line of centers called the homothetic center.

Both *College Geometry* by Altshiller-Court and *Advanced Euclidean Geometry* by Johnson have homothecy for their second chapter, so review this section and read that chapter during the summer to prepare. Also, *Geometry–Do* has an appendix on preparing for *College Geometry*.

Altshiller-Court's ninth chapter (out of ten) is about circle inversion while Johnson's third chapter is this 2nd degree black-belt topic – hopefully, it is *College Geometry* that you will be assigned.

Lemma 5.6

Given $\overline{EFG}$ and P on the circumcircle, ω, long of *$\angle E$, construct the Wallace line determined by P. Extend $\overrightarrow{PP_E}$ to intersect ω at K. Then, $\angle KP_EP_F = \angle PGE$.*

Proof for the Acute Case

Suppose $\overline{PE}$ is a diameter of ω. By Thales' diameter theorem, $\angle KP_EP_F = \rho = \angle PGE$.

K is between E and G; if it is not, re-label F and G. $\overline{PP_EGP_F}$ is cyclic by either the cyclic quadrilateral theorem converse or by Thales' diameter theorem; thus, $\angle GP_EP_F = \angle GPP_F$ by the inscribed angle theorem. $\angle KP_EP_F = \rho \pm \angle GP_EP_F$. $\angle PGE = \rho \pm \angle GPP_F$ by the exterior angle theorem. Subtract ρ and substitute equals. •

The **Steiner line** is parallel to the Wallace line such that the Wallace line is halfway between it and the Wallace line's pedal point. L. 5.7 proves that H is on it, so it is also called the ortholine.

Lemma 5.7

Onto the lemma 5.6 figure, construct the Steiner line; let S be its intersection with $\overleftrightarrow{PK}$. Let the altitude from E intersect the Steiner and Wallace lines at H and L, respectively. Extend $\overrightarrow{EE'}$ to intersect ω at H''. Then, H is the orthocenter of $\overline{EFG}$ and M_{HP} is on the Wallace line.

Proof for the Acute Case

By the transversal theorem corollary (right angles at E' and P_E), $\overleftrightarrow{HE'} \parallel \overleftrightarrow{SP_E}$. By the parallelogram theorem, $\overline{HSP_EL}$ is a parallelogram. By the inscribed angle theorem, $\angle PKE = \angle PGE$. By lemma 5.6, $\angle KP_EP_F = \angle PGE$. By the transversal lemma, $\overleftrightarrow{KE} \parallel \overleftrightarrow{P_EL}$. By the parallelogram theorem, $\overline{KELP_E}$ is a parallelogram. By transitivity, $\overline{HS} = \overline{EK}$. $\overline{H''P} = \overline{EK}$ by the parallels and circle theorem; so, $\overline{HS} = \overline{H''P}$. By the isosceles triangle frustum theorem, $\overline{HH''PS}$ is a triangle frustum, so $\angle SHE' = \angle PH''E'$. By mid-segment theorem #2, $\overline{P_ES} = \overline{P_EP}$. By SAS, $\overline{E'P_ES} \cong \overline{E'P_EP}$, so $\overline{E'S} = \overline{E'P}$. By ASL, $\overline{HSE'} \cong \overline{H''PE'}$, so $\overline{HE'} = \overline{H''E'}$. By the orthocenter and circumcircle theorem, H is the orthocenter of $\overline{EFG}$, H'' is H_{FG} and, by mid-segment theorem #2, M_{HP} is on the Wallace line. •

Wallace Lines and Euler Circle Theorem

The two Wallace lines determined by the endpoints of a diameter of a triangle's circumcircle are perpendicular and intersect on the Euler circle.

Proof

Let P and Q be endpoints of a diameter of the circumcircle. By Wallace theorem V, the Wallace lines determined by P and Q are perpendicular. Draw lines through P and Q parallel to their Wallace lines; these too are perpendicular. By Thales' diameter theorem, their intersection, Y, is on the circumcircle. By lemma 5.7, M_{HP} is on the Wallace line determined by P, so, by mid-segment theorem #2, M_{HY} is on the Wallace line determined by P. Analogously, M_{HY} is on the Wallace line determined by Q. So M_{HY} is on both Wallace lines – it is their intersection – and it is midway from the orthocenter to a point on the circumcircle. By the Euler bisection theorem, it is on the Euler circle. ■

We will here define the **2nd Torricelli point**, V, the **2nd Torricelli apexes** and the **2nd Torricelli segments**. If the triangle is too obtuse, we do not use these terms.

Second Torricelli Lemma *The 2nd Torricelli segments are concurrent at the 2nd Torricelli point.*

Second Torricelli Angles Theorem *If V is long of $\angle EFG$, $\angle EVG = 2\varphi$ and $\angle EVF = \angle FVG = \varphi$.*

Torricelli Points and Euler Circle Theorem
The midpoint of the two Torricelli points is on the Euler circle.

Proof

Given $\overline{EFG}$, relabel E, F, G so V is long of $\angle EFG$. Let M_{EU}, M_{FU}, M_{GU} be the midpoints between the vertices and the 1st Torricelli point, U. The Euler circles of $\overline{EUF}$ and $\overline{EUG}$ intersect at M_{EU} and another point we will call M. In the former circle, by the inscribed angle theorem, $\angle M_{EU}MM_{GE} = \angle M_{EU}M_{GU}M_{GE}$. In $\overline{EUG}$, by mid-segment theorem #1, $\overleftrightarrow{EM_{EU}} \parallel \overleftrightarrow{M_{GU}M_{GE}}$ and $\overleftrightarrow{M_{EU}M_{GU}} \parallel \overleftrightarrow{M_{GE}E}$; thus, by the parallelogram theorem, $\overline{EM_{EU}M_{GU}M_{GE}}$ is a parallelogram and $\angle M_{EU}M_{GU}M_{GE} = \angle UEG$. By transitivity, $\angle M_{EU}MM_{GE} = \angle UEG$. Analogously, $\angle M_{EU}MM_{EF} = \angle UEF$. Adding these two equalities gives us $\angle M_{EF}MM_{GE} = \angle E$. $\angle E = \angle M_{EF}M_{FG}M_{GE}$ by medial triangle theorem I; by transitivity, $\angle M_{EF}MM_{GE} = \angle M_{EF}M_{FG}M_{GE}$. By the inscribed angle theorem converse, $M_{EF}, M_{FG}, M, M_{GE}$ are concyclic; that is, M is on the Euler circle of $\overline{EFG}$. Define M'' so $\overline{GM''E}$ is the homothetic double of $\overline{M_{GU}MM_{EU}}$ with homothetic center U; so, $M \equiv M_{UM''}$. $2\varphi = \angle M_{EU}UM_{GU} = \angle M_{GU}M_{GE}M_{EU} = \angle M_{GU}MM_{EU} = \angle GM''E$. The equalities are by the Torricelli angles theorem, parallelogram angles theorem (mid-segment theorem for $\overline{UM_{GU}M_{GE}M_{EU}}$ a parallelogram), inscribed angle theorem and homothecy, respectively. By the second Torricelli angles theorem and the inscribed angle theorem converse, M'' is on the circumcircle of $\overline{GVE}$. Analogously, M'' is on the circumcircles of $\overline{EVF}$ and $\overline{FVG}$. Thus, $M'' \equiv V$. By homothecy, M (on the Euler circle of $\overline{EFG}$) is the midpoint of $\overline{UV}$. ■

By tying together the Wallace lines, Torricelli points and the Euler circle, these two theorems conclude the red-belt chapter. When black belts learn of harmonic division, they will prove the Feuerbach theorem, which relates the Euler circle to the incircle and the three excircles.

But now, problems! Red belts cannot hope to win the *International Mathematical Olympiad* – a *Yi–Dan* black belt might – but you can at least avoid having to hand in a blank exam paper.

Problem 5.8 *Let ω_1 and ω_2 with centers O_1 and O_2 have common point E. Let F and G be points on ω_2. Also, let $F'' := \omega_1 \cap \overleftrightarrow{EF}$ and $G'' := \omega_1 \cap \overleftrightarrow{EG}$. Prove $\overleftrightarrow{FG} \parallel \overleftrightarrow{F''G''}$.*

> ***Case One: Disjoint Circles***
> By the common point theorem, E is on the line of centers. By the tangent theorem, there is a line, ℓ, through E perpendicular to $\overleftrightarrow{O_1O_2}$. Let $H := \overleftrightarrow{FG''} \cap \ell$ and $H'' := \overleftrightarrow{F''G} \cap \ell$. $\angle HEF = \angle EGF$, by the tangent and chord theorem. Analogously, $\angle H''EF'' = \angle EG''F''$. But $\angle HEF = \angle H''EF''$ by the vertical angles theorem, so $\angle G''GF = \angle F''G''G$. By the transversal lemma, $\overleftrightarrow{FG} \parallel \overleftrightarrow{F''G''}$. •

Proving this for one circle inside the other is left as an exercise.

Lemma 5.8
Given the base and the orthocenter, if it is not on the base, a triangle is fully defined.

> ***Proof***
> Given $\overline{EF}$ and orthocenter H, construct $\overleftrightarrow{EH}$ and $\overleftrightarrow{FH}$. Let E' and F' be the feet of perpendiculars dropped onto $\overleftrightarrow{EH}$ from F and onto $\overleftrightarrow{FH}$ from E, respectively. By the perpendicular length theorem, E' and F' are unique, so $\overleftrightarrow{EF'}$ and $\overleftrightarrow{FE'}$ are fully defined and their intersection, G, is unique. Thus, $\overline{EFG}$ is fully defined. ■

If the triangle is right, then E' and F' will coincide with H and, thus, this point is G. It is also true that, given the base and either the incenter or the medial point, a triangle is fully defined. But this is not true if given the circumcenter. Do you see why?

Problem 5.9 *Let ω_1 and ω_2 with centers O_1 and O_2 have common chord $\overline{EF}$. Let $J := \overleftrightarrow{O_1F} \cap \omega_1$ and $M := \overleftrightarrow{O_1F} \cap \omega_2$ and $K := \overleftrightarrow{O_2F} \cap \omega_1$ and $L := \overleftrightarrow{O_2F} \cap \omega_2$. (Assuming J, M, K, L exist.) Prove:*

1. *$\overrightarrow{JK}, \overrightarrow{EF}, \overrightarrow{LM}$ are concurrent at a point P.*
2. *J, E, M, P are concyclic.*
3. *L, E, K, P are concyclic.*

Solution

1. By Thales' diameter theorem, $\angle JEF$ and $\angle LEF$ are right, so J, E, L are collinear. Let $P := \overrightarrow{JK} \cap \overrightarrow{EF}$ and $P'' := \overrightarrow{LM} \cap \overrightarrow{EF}$. By Thales' diameter theorem, $\angle K$ and $\angle M$ are right, so, by the orthocenter theorem, F is the orthocenter for a triangle with base $\overline{JL}$. By lemma 5.8, $\overline{JLP} \cong \overline{JLP''}$ and P is P''.

2. By the inscribed angle theorem, $\alpha = \angle FEM = \angle FLM$. By Thales' diameter theorem, $\rho = \angle E = \angle K$. Let $\beta = \angle KPL$. $\angle PKL = \rho$, so $\alpha = \rho - \beta$. $\angle JEM = \rho + \alpha = \sigma - \beta$. By the cyclic quadrilateral theorem converse, J, E, M, P are concyclic.

3. Analogously, L, E, K, P are concyclic. ■

Problem 5.10 *Prove that the orthocenter of a triangle is the incenter of the triangle whose vertices are where the given triangle's altitudes cut its circumcircle.*

Solution to Acute Case

By the orthocenter and circumcircle theorem, the altitudes cut the circumcircle of $\overline{EFG}$ at the orthic reflections, H_{EF}, H_{FG}, H_{GE}. By mid-segment theorem #1, $\overleftrightarrow{H_{EF}H_{FG}} \parallel \overleftrightarrow{G'E'}$ and $\overleftrightarrow{H_{FG}H_{GE}} \parallel \overleftrightarrow{E'F'}$ and $\overleftrightarrow{H_{GE}H_{EF}} \parallel \overleftrightarrow{F'G'}$. By the orthic triangle incenter theorem, the orthocenter of $\overline{EFG}$ is the incenter of $\overline{E'F'G'}$ and, since the sides of $\overline{H_{EF}H_{FG}H_{GE}}$ and $\overline{E'F'G'}$ belong to pairwise parallel lines, angle bisectors of $\overline{E'F'G'}$ are angle bisectors of $\overline{H_{EF}H_{FG}H_{GE}}$. Thus, their incenters concur at the orthocenter of $\overline{EFG}$. •

Observe that $\overline{H_{EF}H_{FG}H_{GE}}$ is the homothetic double of $\overline{E'F'G'}$ with homothetic center H. This is analogous to how the double long triangle is the homothetic double of the long triangle with homothetic center I, but P. 5.10 is a different pair of triangles and a different center, so students should be careful not to conflate them. If you close your good eye, the figures look much alike.

The next problem introduces some terminology that red belts need to know. The **tangential triangle** is an example of an **antipedal triangle**. Since the pedal triangle is formed by dropping perpendiculars to the sides and connecting their feet, the antipedal triangle is formed by drawing segments from the pedal point to the vertices and perpendiculars through them at the vertices to form the sides. If we take the circumcenter as the pedal point, then, by the tangent theorem, the antipedal triangle sides are tangent to the circumcircle; hence, its name.

Problem 5.11 *Prove that the orthic triangle and the tangential triangle are homothetic and that their homothetic center is on the Euler line, but that it is not the orthocenter.*

This problem shows there are two ways to form the tangential triangle; either directly from the circumcenter as described in the paragraph above, or indirectly from the orthic triangle. Which is easier depends on whether you have already found the circumcenter or have already found the orthic triangle. The following problem provides a third way to find the parent triangle.

Problem 5.12 *Prove that the parent triangle is antipedal if the orthocenter is the pedal point.*

1. Draw the three medians and extend them outside the triangle a distance equal to the segment inside the triangle. Connect the endpoints.

2. Draw lines through each vertex parallel to the opposite sides and cut them off a distance from the vertex that is equal to the length of the opposite side. Connect the endpoints.

3. Draw segments from the orthocenter to the vertices and perpendiculars through them at the vertices to form the sides of the parent triangle. Cut them off where they intersect.

The first works well with a center-finding ruler like the *Geometry–Do* ruler; the second requires drawing parallel lines and then laying off lengths on them, which the *Geometry–Do* ruler can do, though this is slower than the first method. The third is fastest, but only if one already has the orthocenter; it can be done with the *Geometry–Do* ruler or any clear plastic right triangle.

Yellow belts learned that the contact triangle is the pedal triangle if the incenter is the pedal point. In the following problem we will show that the double-long triangle – also known as the excentral triangle – is the antipedal triangle when the incenter is the pedal point.

Problem 5.13 *Prove that the double-long triangle is antipedal if the incenter is the pedal point.*

The following triangle centers all have named triangles for their pedal or antipedal triangles.

Pedal Point	Pedal Triangle	Antipedal Triangle
Circumcenter	Medial Triangle	Tangential Triangle
Incenter	Contact Triangle	Double-Long Triangle
Orthocenter	Orthic Triangle	Parent Triangle

Problem 5.14 *Given $\overline{EFG}$ with E, F, G counterclockwise, find:*
1. *P such that $\angle PEF = \angle PFG = \angle PGE$. This angle is α; P is the **first Brocard point**.*
2. *Q such that $\angle QEG = \angle QGF = \angle QFE$. This angle is β; Q is the **second Brocard point**.*
3. *Prove that $\alpha = \beta$. This is called the **Brocard angle**.*

E, F, G counterclockwise is standard procedure in geometry, though this practice is not always followed. In this problem, if they were clockwise, it would switch the first and second Brocard points. The **Brocard triangle** is formed by the intersection of segments from the vertices to their associated Brocard points. It is inscribed in the **Brocard circle**, which is related to the **symmedian**. Henri Brocard was a French naval officer and a meteorologist; but it was an Englishman, Lewis Fry Richardson, who devised the seven axioms of meteorology, giving it the rigor of geometry.

A discussion of Brocard's work and of **complete quadrilaterals** could conceivably go in the red-belt chapter because they are advanced triangle geometry, though some complete quadrilateral theorems involve the Newton line, which is quadrature theory. However, I have decided to put this material in the *Teacher's Manual* as challenge problems for gifted students. It is not generally taught in the high schools of any country and the aspiring engineer who has only three years to study geometry in high school is advised to press on from here to quadrature theory (blue belt), harmonic division (*Cho–Dan*, 1st degree black belt), circle inversion (*Yi–Dan*, 2nd degree black belt), and, if there is time, to projective geometry (*Sam–Dan*, 3rd degree black belt).[98]

Sam–Dan is the highest rank that one can obtain; the *Sam–Dan* of *Geometry–Do* has enough background to read Mihalescu or any of several other advanced geometry textbooks that are available in English. They are all different and they each have their followers; the student should ask the professors in his chosen specialty for guidance in any further study of geometry. Mihalescu was an artillery officer during WWII and discusses problems of interest to army officers as well as to engineers; aspiring mathematicians may want a more abstract textbook that discusses non-Euclidean geometry, topology, and other topics in pure mathematics.

The *Yi–Dan* of *Geometry–Do* probably has enough geometry for the *International Mathematical Olympiad*, with the possible exception of geometry on the complex plane, which we cannot get into because complex analysis is an upper-division college course, and this is a high-school textbook. The IMO also asks questions about number theory and combinatorics. The teenage mathematician should keep his interest in mathematics broad; there will be plenty of time to choose a specialty before becoming a PhD.[99]

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[98] Thus, the recent boldface terms (from P. 5.14 onward) do not appear in the glossary. In keeping with my policy of not defining terms unless they are used later, Brocard's work, the symmedian and complete quadrilaterals will be left to advanced students and their teachers – with the help of the *Teacher's Manual* – to research on their own. I do not want to be like *Common Core* that is always defining terms and then never using them. This just results in geometry being reduced to a long and boring vocabulary test of words that the students will never use – like learning a language that is spoken in a country that one will never visit. *Geometry–Do* is all about practical applications.

[99] *The International Journal of Computer Discovered Mathematics* presents many geometric results that remain unproven and are left as exercises. Proving any one of them will get you published in a refereed journal while you are still a teenager. www.journal-1.eu Try! "Published author" looks really good on a college application. 😉

Red Belt Exit Exam

1. *Problem 4.14 can be solved by citing the long quadrilateral theorem; please do so.*

2. *Prove the incenter, circumcenter and bi-medial of a bi-centric quadrilateral to be collinear.*

3. *Let* $\overline{EFG}$ *be acute,* $\overleftrightarrow{F'G'} \cap \omega = \{P_1, P_2\}$ *and* $Q_1 := \overleftrightarrow{FP_1} \cap \overleftrightarrow{E'G'}$ *and* $Q_2 := \overleftrightarrow{FP_2} \cap \overleftrightarrow{E'G'}$*. Prove* P_1, P_2, Q_1, Q_2 *are concyclic.* $\{\,\}$ *denotes a set, so* P_1 *and* P_2 *are the two intersections.*

4. *Given* $\overline{EFG}$ *with incenter* I *and excenters* X, Y, Z*, prove that any two of these centers and the two triangle vertices that are not collinear with them are concyclic. Locate the center.*

5. *Given* $\overline{EFG}$ *with excenters* Y *and* Z*, let* $\overline{EFN_G}$ *be isosceles with* N_G *on the circumcircle of* $\overline{EFG}$ *and on the same side of* $\overleftrightarrow{EF}$ *as* G*. Prove that* E, F, Z, Y *are concyclic with center* N_G*.*

6. *Prove that the Carnot theorem is true for obtuse triangles, except that one must subtract the perpendicular dropped from the circumcenter onto the side opposite the obtuse angle.*

7. *Given* $\overline{EFGH}$ *and* $P := \overrightarrow{EF} \cap \overrightarrow{HG}$ *and* $Q := \overrightarrow{FG} \cap \overrightarrow{EH}$*, prove that the circumcircles of* $\overline{FGP}$ *and* $\overline{GHQ}$ *intersect on* $\overline{PQ}$ *if and only if* $\overline{EFGH}$ *is cyclic.*

8. $\overline{EFGH}$ *is cyclic and orthodiagonal with bi-medial* T*. Prove* $\overline{T_{EF}T_{FG}T_{GH}T_{HE}}$ *to be bi-centric.*

9. *Demonstrate elementary quadrature theory by proving the diagonal bisection theorem.*

Note that problem #2 is due to none other than the famous physicist, Isaac Newton. He proved what physicists call their superb theorem[100], that a spherically symmetric mass distribution attracts a body outside it as if its entire mass were concentrated at its center. Chandrasekhar[101] said that Newton's geometric proof in *Principia,* "must have left its readers in helpless wonder."

The editor[102] of the *Real-World Economics Review* writes, "It is a completely mistaken idea that scientific theory is based on deductions from a series of postulates." This is not true. The *World Economics Association*, aka the *Post-Autistic Economics Network*, is wrong. They sow hatred for logic while promoting their Marxist agenda by writing every sentence, "Statistics show that [Marxist dogma]." How clever! Marxism has had a 150-year streak of bad luck with logic. *Bums!*

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[100] ***Newton's Superb Theorem: An Elementary Geometric Proof*** by Chris Schmid: arxiv.org/abs/1201.6534

[101] At the age of 19 he deduced the maximum mass of a stable white dwarf star. More massive white dwarves can – if they do not throw off mass – collapse with enough gravitational pressure that it overcomes the electron degeneracy pressure, so their atomic nuclei are pressed together into one big lump, which creates a neutron star.

[102] www3.unifr.ch/econophysics/?q=content/deification-science-its-disastrous-consequences

Practice Problems

5.15 *Given* $\overline{EFG}$ *with incenter* I *and* $\angle G$ *equal to* φ. *If* $\overrightarrow{EI}$ *and* $\overrightarrow{FI}$ *intersect the opposite sides at* J *and* K, *respectively, prove that* $\overline{IJ} = \overline{IK}$.

5.16 *Given a circle, find a point on the extension of its diameter from which the tangents are equal to the radius.*

5.17 *On each leg of a right triangle, find points equidistant from the hypotenuse and the apex.*

5.18 *Given a circle between two parallel lines, draw a tangent that is cut by the lines at a length equal to a given segment.*

5.19 *Construct a right triangle given the hypotenuse and the length from its midpoint to a leg.*

5.20 *Construct a triangle given an altitude and the circumradii of the two triangles it divides.*

5.21 *Given the orthic triangle,* $\overline{E'F'G'}$, *construct its associated triangle,* $\overline{EFG}$.

5.22 *Given the radius, draw a circle with its center on a line and touching another circle.*

5.23 *Given a circle, a line outside it and an angle, circumscribe a triangle with an interior angle equal to the given angle and that vertex on the line.*

5.24 *Given two points and two segments, draw a circle with radius equal to one segment that passes through one point and has tangents to the other point equal to the other segment.*

5.25 *If two circles are tangent at* G *and* $\overleftrightarrow{EF}$ *is tangent at* E *and* F, *prove that* $\angle EGF$ *is right.*

5.26 *Given* $\overline{EFG}$ *with* P *on the circumcircle between* E *and* F, *prove that the Wallace line with respect to* P *bisects* $\overline{HP}$.

5.27 *Prove that the Wallace line determined by* P *intersects the Euler circle at* M_{PH}.

5.28 *Given* $\overline{EFG}$, *let* $\overline{JKL}$ *have sides that go through* E, F, G *and extensions parallel to the opposite sides of* $\overline{EFG}$. *Prove that the circumcircle of* $\overline{EFG}$ *is the Euler circle of* $\overline{JKL}$.

5.29 *Through one of two points of intersection of two circles, draw a line so the circles cut off two chords, one three times the length of the other.*

Isometric Transformations without Linear Algebra

Defining geometry as "properties of geometric figures that are not changed by motion" makes sense only to those who know linear algebra; without vectors, motion is nowhere defined. Students are profoundly frustrated by *Common Core* geometry, largely because of its practice of saying "slide this figure over there" or "rotate this figure around that point." If the students know anything about geometry, they know that the pencil lead is impregnated in the paper. Geometric figures cannot be moved anywhere; they can be redrawn elsewhere on the paper, but that is all.

To translate a segment is to draw another segment parallel to the given segment and equal to it in length. By the equal segments on parallels theorem, this forms a parallelogram. Thus, saying that you are translating a segment is just another way of saying that you are drawing a parallelogram. Nothing is gained by teaching students two ways of saying the same thing.

Analogously, saying that you are rotating a point is just another way of saying that you are drawing an isosceles triangle with a given apex angle. And rotating a line around a point on it is another way of saying that you are laying a given angle against the line. Claiming that all of geometry is about isometric transformations is not profound, it is just a different phraseology. And, frankly, it is venal because, as soon as the definition of geometry as "properties of geometric figures that are not changed by motion" has been invoked, the teacher has become a shill for Bill Gates. Motion of geometric figures does not exist on a sheet of paper; it exists only on the screen of computers that are running animation software already in use in shooter video games, except now ten times the price – *The taxpayers will pick up the tab!* – and boring because it is triangles, not horrific mutants, that are moving around the screen. And you do not even get to shoot them.

Nevertheless – *No motion involved!* – I will explain how to rotate a line around a point not on it.

To rotate a line around a point not on it is to construct a right kite with its definitional diagonal the perpendicular dropped from the point to the line and extended to a length, d, that depends on the angle of rotation. Its midpoint is the circumcenter of the right kite. The pair of sides bracketing the point are the intersection of the circumcircle with the circle around the given point that is tangent to the given line. The other pair of sides extended is the rotation; we always rotate both ways. If the angle of rotation, θ, is given in degrees, then $d = r\, sec\, \theta$ where r is the distance from the point to the line; that is, the radius of rotation. $d = r\, sec\, 60° = 2r$ for a $60°$ rotation; though, for this angle, we already knew that the hypotenuse of a half equilateral triangle is twice the length of the leg at that vertex. This knowledge makes construction of the right kite easy in problem 5.30, next. (If the highways are equally spaced, then the solution is easy, so we will assume that they are not.) In problem 5.31, we will demonstrate how to construct the right kite that defines rotation when we must use C. 1.5 to replicate the angle of rotation.

Problem 5.30 *A nuclear power plant is to be built in an area where there are three parallel highways. There is concern that terrorists might attack, either by land from one of the highways, or by air. Military posts will be built on each of the highways with infantry that can respond to a ground attack. Also, these posts are to be at the vertices of an equilateral triangle with the power plant at the center, so their anti-aircraft guns provide uniform coverage of the air space above it.*

Solution

Let E, F, G be the vertices of the equilateral triangle with F on the middle highway. Since there are no conditions restricting the lateral position of the triangle on the highways, we can choose G arbitrarily. By the angle sum theorem, the interior angles of an equilateral triangle are all φ, so, if the middle line passes through F, then a rotation of it by this angle will pass through E, which we know to be on the E-line.

Drop a perpendicular from G to the middle line and call this length r. Extend it another r so the definitional diagonal is $2r$ long. Around its midpoint draw a circle of radius r; by Thales' diameter theorem, this diameter subtends a right angle at any point on the circle. Draw a circle of radius r around G. It intersects the first circle at points that are the right vertices of right triangles with one leg half the diameter; that is, half equilateral triangles.

The other pair of sides of this right kite (not r long) are on the middle line rotated by φ; extend them to the E-line and call these points of intersection E_1 and E_2. Build equilateral triangles on the bases $\overline{E_1G}$ and $\overline{E_2G}$, each on the other side of the right vertex of their respective half equilateral triangles. Their apexes should be on the F-line. ■

Problem 5.31 *Given an angle, two lines, and a point between them, draw a circle around the point so the lines cut off a chord that subtends at the center the given angle.*

Solution

Drop a perpendicular from the point, O, onto the nearer line, with foot O'. On one side of $\overrightarrow{OO'}$, by C. 1.5, replicate the given angle and lay off the distance from the point to the line so $\overline{OO''} = \overline{OO'}$. Find E on the second line such that $\overline{EO''} \perp \overline{OO''}$. Draw a circle around O of radius $\overline{OE}$ and let F be its intersection with the first line so E and F are on the same side of $\overleftrightarrow{OO'}$. $\overline{EF}$ subtends the given angle at the center of the O-circle. ■

Replicating on the same side of $\overleftrightarrow{OO'}$ as the line intersection fails if the given angle and the angle made by the line intersection are supplements; on the other side, it fails if they are equals. Thus, there are one or two solutions unless the line intersection and the given angle are both right. This fails unless the point is on the angle bisector of the lines, in which case you draw a square.

On the Difference Between Engineering and Competition Problems

Engineers are always constrained by conditions beyond their control. For instance, if you are contracted to build a road between two towns that are divided by a river, the bridge must be roughly between them; you cannot expect people to add an hour to their journey just because you found a better site for bridge building fifty klicks downstream. Military engineers must meet even more stringent constraints. The generals have the final say and they know what they are doing when it comes to making use of terrain features. This is why many defensive positions are built without any input from the engineers; there was little leeway in where to position weapons and, in haste, their positioning was just not optimized. But an engineer with a compass and straightedge can optimize the design in an hour. Haste is no excuse for being less than optimal!

Suppose there are two straight roads that intersect behind enemy lines at $45°$ and a rock outcropping between them, on your side. You will blast out a cave in the front of the rock and pour concrete for a bunker with a $30°$ slit to fire an automatic cannon through. The enemy comes, and you will probably be taking fire while the concrete is still green. *You must hurry!!!*

You do not know which road the enemy will choose and it takes time and hurts accuracy if you must adjust your sights as they approach, so it is best if you know of points on each road equally distant from your gun; also, if these points are on the edges of your gun's top traverse, then the enemy is as close as possible[103] and the gunner can quickly locate his kill zone by transversing to the edge of his bunker slit. So, how do you pour the concrete for your bunker slit, optimally?

It is problem 5.31!!! Did you get it? P. 5.31 is an engineering problem because there are only a few well-described cases when the construction fails – when the top traverse of the gun and the angle made by the roads are equal or supplementary – and, since these are external constraints of independent origin, it is highly unlikely that they will happen to cause failure of the geometry.

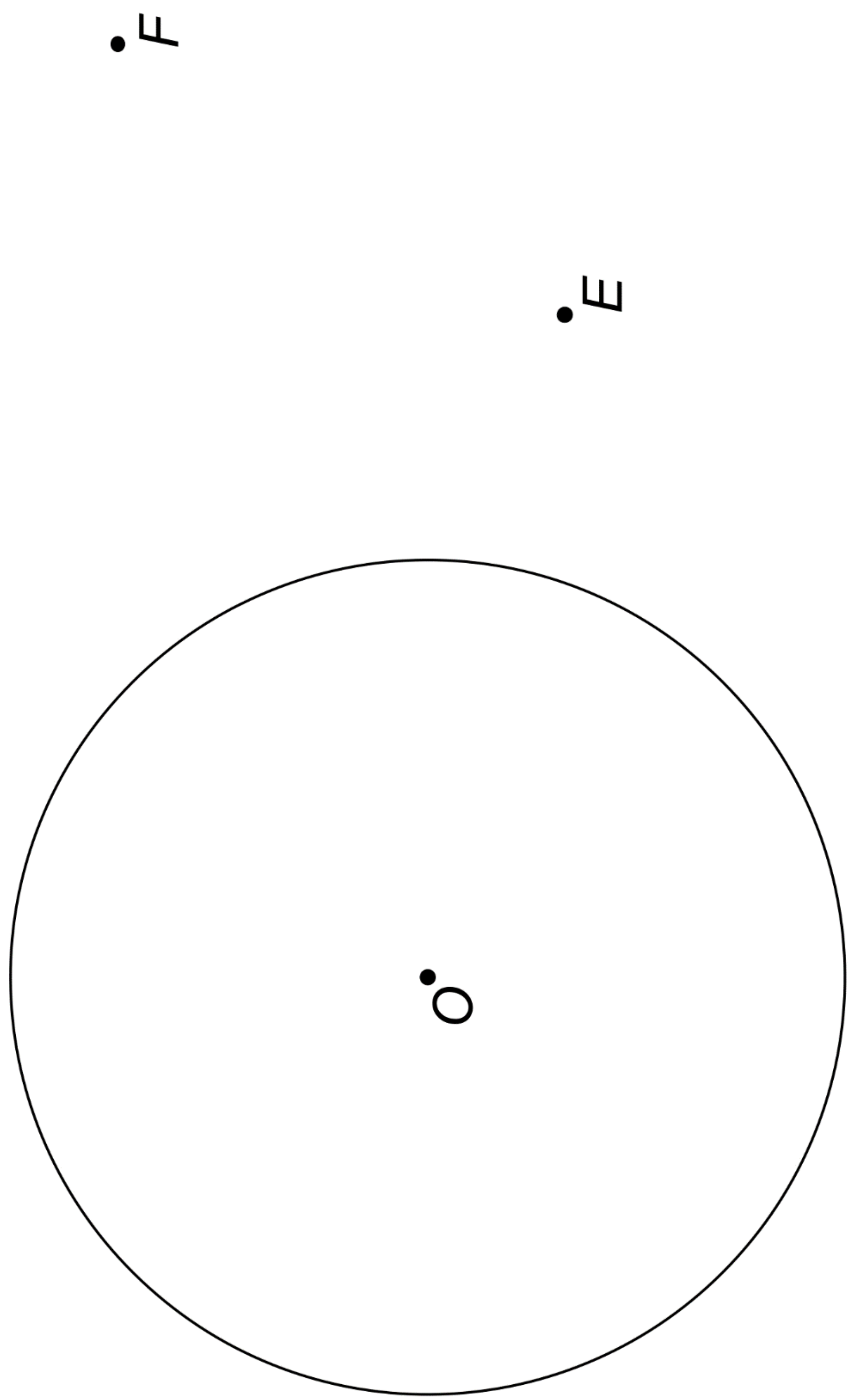
In mathlete competitions, the problem exists untethered to terrain or tactical considerations. If it says, "diagram not drawn to scale," this is a big clue that, if it were, you could just measure the asked-for length or angle with your ruler or protractor. So, draw it yourself! But, if the diagram is drawn accurately, this too is a big clue. It means that the construction is probably not useful to engineers because it almost never works for arbitrarily chosen parameters. The exam was composed by drawing the solution and then just deleting some of the lines, so it is problematic.

In problem 5.32, I drew two parallel lines through the circle, picked exterior points on them, and measured the "given" angle. Then I deleted the lines to make it problematic. Do this in reverse!

[103] You have heard of KISS; Keep It Simple, Stupid? I teach KESS; Killing's Easier at Short-Range, Stupid.

Problem 5.32

In the figure shown, find points G and H on the circle such that the chord $\overline{GH}$ subtends at the center an angle of φ and such that $\overleftrightarrow{FG} \parallel \overleftrightarrow{EH}$.



On the Relation Between Geometry and Probability

Common Core mandates that geometry classes include probability/statistics. In the introduction to the *Glossary*, I mock *Glencoe Geometry* for wasting time doing statistics on how many ladies visit a hair salon. ROFL!!! Amateur sociologists – Isn't everybody one? – teaching in high school just *love* this about *Common Core* because it means that they can spend the year teaching basic statistics – the only math they know – and yet, inexplicably, draw pay as a "geometry" teacher. Geometry has nothing to do with introductory statistics, but quadrature *is* related to probability.

All aspiring engineers learn integral calculus: finding area. They are learning about Reimann integrals, which are over bounded intervals on the real number line – think of all the skinny rectangles standing shoulder to shoulder – which is later extended to unbounded intervals out to infinity. Such integrals may not be defined, but students learn which ones are; for example, the integral of $\frac{1}{x}$ out to infinity is infinite, but the integral of $\frac{1}{x^n}$ out to infinity is finite if $1 < n$.

$[a, b]$ is the set of all real numbers x such that $a \leq x \leq b$, and (a, b) is the set of all real numbers x such that $a < x < b$. The Reimann integral of $f(x)$ over $[a, b]$ or (a, b) is the same thing, even if $f(a)$ or $f(b)$ do not exist. For instance, $\int_0^\infty \frac{1}{x^2} dx$ over $[0, \infty)$ is undefined because $\frac{1}{0^2}$ is undefined, but you can just ignore the unpleasant $x = 0$ and integrate over $(0, \infty)$. Analogously, you can ignore holes in a function when integrating it. If $f(x) = x$ for all x except $x = 1$, where $f(1) = y$, with y any real number or even if it is just undefined, $\int_0^2 f(x)dx = 2$, regardless of what y is. There is an infinity of points in $[0, 2]$, and a finite number of weird ones do not change the integral over $[0, 2]$. Lebesgue integration, which Kolmogorov used to give probability an axiomatic foundation, takes this discussion even further to consider integrating over sets more complicated than just intervals. That Kolmogorov was an axiomatist gives relation to probability and geometry; though, that Kolmogorov was associated with the *New Math* that failed so miserably in America forty years ago tarnishes his legacy. Set theory in elementary school is age inappropriate and unneeded. (*Common Core* is making a similar mistake by teaching tykes tricks used by 1950s accountants to speed up doing their sums; age inappropriate and unneeded!)

The important point, regarding Kolmogorov's theory of probability, is that it is possible for an event to have zero probability even if particular cases are possible, provided that these cases are a finite subset of an infinite event space. For instance, if $f(x) = x$ for all x except $x = 1$, where $f(1) = 5$, then the probability of $f(x) = 5$ for $x \in [0, 2]$ is zero. Seriously. One exception out of an infinity does not change the probability. It is not *almost* zero probability; it is just plain zero.

Disjoint Figures that do not overlap; their areas form an additive group
This includes touching circles and adjacent triangles, if outside each other.

This definition may confound some readers. It may seem that two circles that touch are not disjoint because they have a common point. Indeed, "nothing in common" is the dictionary definition of disjoint, and it is the colloquial meaning of the word. But mathematical terms are not always in line with their colloquial meaning. It is important to us that the areas of disjoint figures form an additive group; without group theory, quadrature theory has no foundation!

Euclid said, "a point is that which has no part, and a line is breadthless length," which has left a lot of people asking, "What?" This statement can best be understood in the context of probability theory. The probability of $f(x) = 5$ for $x \in [0, 2]$ is zero for the above function because points have no length. If $\overline{EFG} \cong \overline{JKL}$, then $\left|\overline{EFGH}\right| = \left|\overline{EFG}\right| + \left|\overline{GHE}\right| = \left|\overline{JKL}\right| + \left|\overline{GHE}\right|$; that is, the total area of figures is the same whether they are adjacent or drawn away from each other. The probability of a point inside $\overline{EFGH}$ being on $\overline{EG}$ is zero because lines have no area.

In the preceding section, *On the Difference Between Engineering and Competition Problems*, I write that P. 5.32 is, "probably not useful to engineers because it almost never works for arbitrarily chosen parameters." Mathematicians may have thought that I had gone soft, using a vague, colloquial term like "probably." But, while common men do use this term frequently and often in the vaguest sense, it *does* have a rigorous mathematical definition.

Problem 5.31, in the military context of the two lines being roads and the given angle being the top traverse of an automatic cannon firing through a bunker slit, is an engineering problem. This is because the probability of the geometry working is 100%. Even though the geometry fails when the angles are equal or supplementary, the probability of this occurring is zero.

Problem 5.32 is not useful to engineers, though it might be good fun at a mathlete competition, because the probability of the geometry working is almost zero in cases where the points are given by tactical considerations such as rock outcroppings, and the angle is given by the top traverse of a gun. The way to make the geometry work for a mathlete exam is to start with the parallel lines, pick some points on them, and then measures the so-called "given" angle. Then, in a step made possible only with tracing paper or the "delete" button on graphic software, remove the parallel lines. The probability of this working for an arbitrary angle is not exactly zero, but there is little choice unless the points are very close to each other relative to the diameter.

Awareness of the two types of problems help one approach an exam. P. 5.32 seems contrived, so assume the problem is solved. Draw parallels through E and F, label G and H and the angle subtended at the center by $\overline{GH}$ as θ. Rotate $\overline{HE}$ around O by θ to $\overline{GJ}$. $\angle FGJ = \theta$ (verify), so G is on the locus of vertices for θ subtended by $\overline{FJ}$. We want φ, so find J by rotating E by φ, then construct the locus arc by P. 4.5. It cuts the circle in $0, 1$ or 2 places. For each G, by C. 3.3, is H.

Disjoint There is zero probability of any points being inside both figures

Measure The size of sets; counts of discrete points, lengths of segments or areas of triangles

Probability The ratio of the measure of a subset to the measure of the whole set

Random The points in a segment or inside a triangle or circle are uniformly distributed

This is *not* how other geometry textbooks define these terms. On the back cover, I say that organization is my principal contribution to geometry, but my effort in *Volume Two* to unite geometry with Kolmogorov's *Foundations of Probability* is original – hopefully in a good way.

Most geometry textbooks define disjoint as figures whose intersection is impossible; that is, there is no intersection. As explained above, this is not the same thing as there being zero probability of any points being inside both figures; they can touch, but just not overlap. Note that random assumes a uniform distribution; I have waxed sarcastic on people assuming uniformity without justification[104], but in geometry it *is* justified. Also note that measure refers only to the size of sets, not of angles. Most geometry textbooks define the measure of an angle in terms of radians, but that is because modern geometry textbooks are also introductory trigonometry textbooks.

Geometry–Do is unique in not teaching any trigonometry, except in a couple of stand-alone appendices. The only angles that we know of, in both *Volume One* and *Volume Two*, are φ, ρ and σ. To assign radians to an angle requires measuring the length of its arc, but this can only be done with calculus. In *Geometry–Do*, length refers only to how long a segment is. And area refers only to the size of a triangle or a union of disjoint triangles, never to circles or to other curves, though we do accept that central angles of ρ and σ quarter and halve a circle, respectively. Now let us illustrate these ideas by solving some simple probability problems; let us also introduce the basic discrete probability distributions that all high school students are expected to know.

Problem 5.33 *Pick a number, any number, between* 1 *and* 10*. What is the chance of it being* 5*?*

The chance of 5 is zero. I did not say "pick a whole number," I said, "pick any number." In this case, probability is a ratio of lengths. Points do not have length, so the numerator's length is zero; segments have length, and the length of $[1, 10]$ in the denominator is nine. $P(5) = \frac{0}{9} = 0$. In discrete probability, measure means a finite number of items, like colored balls in an urn; but, in geometry problems, it means lengths or areas – infinities of uniformly distributed points.

[104] People who know nothing of statistics will consider a sample of some data obviously skewed over $[0, \infty)$, locate the minimum, a, the maximum, b, and claim that it is uniformly distributed over $[a, b]$. "Data driven!" they boast.

Problem 5.34 *If $x \in [0, 5]$, what is the probability of x being closer to 1 than it is to 3?*

> ***Solution***
>
> 2 is midway between 1 and 3, so $x \in [0, 2]$ is closer to 1 than it is to 3. Thus, the ratio of the desired length to the whole length is $\frac{2}{5} = 40\%$. Note that $[0, 2)$ is also 2 long. ■

Problem 5.35 *Three points are at random on a circle. What is the chance they are in a semicircle?*

The choose function, $\binom{n}{r}$, is nCr on scientific calculators. It is how many ways you can pick from a finite set without replacement when order does not matter. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ where the ! symbol means factorial; *e.g.,* $5! = 5 \times 4 \times 3 \times 2 \times 1$ and $\binom{5}{2} = \frac{5!}{2!3!} = \frac{5\times4}{2\times1} = 10$. $(n-r)!$ cancels out much of $n!$. Note that $0! = 1$. Learn to do this manually for when you do not have a scientific calculator, or for when you are only choosing two items. The classic example is a menu:

Problem 5.36 *How many ways can you and your date choose from three appetizers, five entrées and four desserts? You intend to share, so you do not want to both get the same of an item.*

> ***Solution*** $\binom{3}{2}\binom{5}{2}\binom{4}{2} = 3 \times 10 \times 6 = 180.$ ■

If you do not mind getting the same menu items – you are choosing with replacement – it is $3^2 5^2 4^2 = 3600$. A lot more! You must know if you are choosing with or without replacement.

In the first two probability distributions below, the probability of success in a single trial – called a **Bernoulli experiment** – is a constant, p. The classic example is flipping a coin; if success is heads, then $p = 0.5$ and this remains constant no matter how many times the coin is flipped. In r samples, you are looking for x goodies. The probability of this *and* that occurring is the product of their probabilities; the probability of this *or* that occurring is the sum of their probabilities. Thus, the probability of x goodies and $r - x$ badies is $p^x(1-p)^{r-x}$, but there are r choose x ways that these can be sequenced, which is an "or" because only one sequence occurs at a time. x and r retain their same meanings in the two equations below; other textbooks swap meanings.

$P(x) = \binom{r}{x} p^x (1-p)^{r-x}$ **binomial distribution**; goodies in r samples $\mu = rp$

$P(r) = \binom{r-1}{x-1} p^x (1-p)^{r-x}$ **negative binomial**; samples needed to get x goodies $\mu = \frac{x}{p}$

If $x = 1$, the latter is the **geometric distribution**. It is $P(r) = p(1-p)^{r-1}$ and is likely in quality control, where "success" results in immediately stopping the machine and ending the sampling.

The hypergeometric is like the binomial except for sampling from a small collection of n items with g goodies. If the collection is vast, like all the tuna in the sea, approximate with the binomial.

$P(x) = \frac{\binom{g}{x}\binom{n-g}{r-x}}{\binom{n}{r}}$ **hypergeometric**; goodies in r samples $\mu = r\frac{g}{n}$

Problem 5.37 *What is the chance of at least two aces in a five-card draw from a* 52*-card deck?*

Solution $\frac{\binom{4}{2}\binom{48}{3}+\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \approx 4\%.$ ■

What if we bet on the draw needed to get a pair of aces? I pay a dollar for every card you deal me, and you pay me $\$25$ when I get a pair of aces. There are two aces in 26 cards, so you have a slim $\frac{1}{25} = 4\%$ advantage. This is roughly the vigorish of blackjack, so you are a fair guy. Yes?

$P(r) = \frac{\binom{r-1}{x-1}\binom{n-r}{g-x}}{\binom{n}{g}}$ **negative hypergeometric**; samples needed to get x goodies $\mu = x\frac{n+1}{g+1}$

Advantage mine!!! The expected number of cards needed to get two aces is $\mu = 2 \times \frac{53}{5} = 21.2$ cards, so any payout of at least $\$22$ wins for me. I will take your bets as fast as you can deal!

What if the trials are not discrete events but rather just the passage of time as you wait for something? Suppose a liquor store averages twenty customers per hour at its drive-thru window. But the clerk needs to use the bathroom, so he is wondering, what is the chance of at least one customer in the next five minutes? An estimate could be made by thinking of every minute as a "discrete" trial with $p = \frac{1}{3}$ because twenty customers an hour is one every three minutes. So, by the binomial distribution, the probability of getting at least one customer in five minutes is 100% minus the probability of getting none; that is, $1 - \left(\frac{2}{3}\right)^5 \approx 87\%$. A time increment of ten seconds yields $1 - \left(\frac{17}{18}\right)^{30} \approx 82\%$. If the time increment can approach zero, it is a **Poisson process**.

$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ **Poisson distribution**; goodies in a given time period $\mu = \lambda$

Lambda, λ, is the average number of events in the time we are considering; in this case, five minutes, so $\lambda = \frac{20}{12} = \frac{5}{3}$ events. Thus, the chance of the clerk's bathroom break being interrupted is 100% minus the chance of no customers; that is, $1 - e^{-\frac{5}{3}} \approx 81\%$. Here, the 82% estimate is better because two customers in ten seconds must queue; they are not independent. But, if events are independent, like radioactive material emitting neutrons, then it is a Poisson process.

Problem 5.38

There is a rectangular skylight in my otherwise lead-sheathed laboratory. If a cosmic ray passes through the skylight, what is the probability that it is closer to the center than to the edge?

Solution

Draw the diagonals; by "center," we mean the bi-medial. It and two vertices define a triangle. By the medial triangle area theorem, a nested triangle with its apex at the bi-medial quarters the area of its parent triangle. By mid-segment theorem #1, the bases of nested and parent triangles have parallel extensions, so they form a rectangle with a quarter of the area of the window. By the mid-segment theorem, part two, any segment from the bi-medial to the edge of the window is halved where it cuts the small rectangle. Thus, the probability of being closer to the center is 25%. ■

Problem 5.39

Two points are randomly placed on a circle; they are connected to each other and to the center. What is the probability that these segments form an acute triangle?

Solution

Let O be the circle center; P and Q are the two points. For the vertex at O to be acute, Q must be in the semicircle that P is on the center line of. There is a 50% chance of this occurring. By the tangent theorem, the other two vertices are acute, so it is 50%. ■

In the following problem, assume that the area of a circle is πr^2, even though this is a result of trigonometry, not geometry. This classic problem is on the border between the two sciences.

Problem 5.40

Two points are randomly placed inside a circle; they are connected to each other and to the center. What is the probability that these segments form an acute triangle?

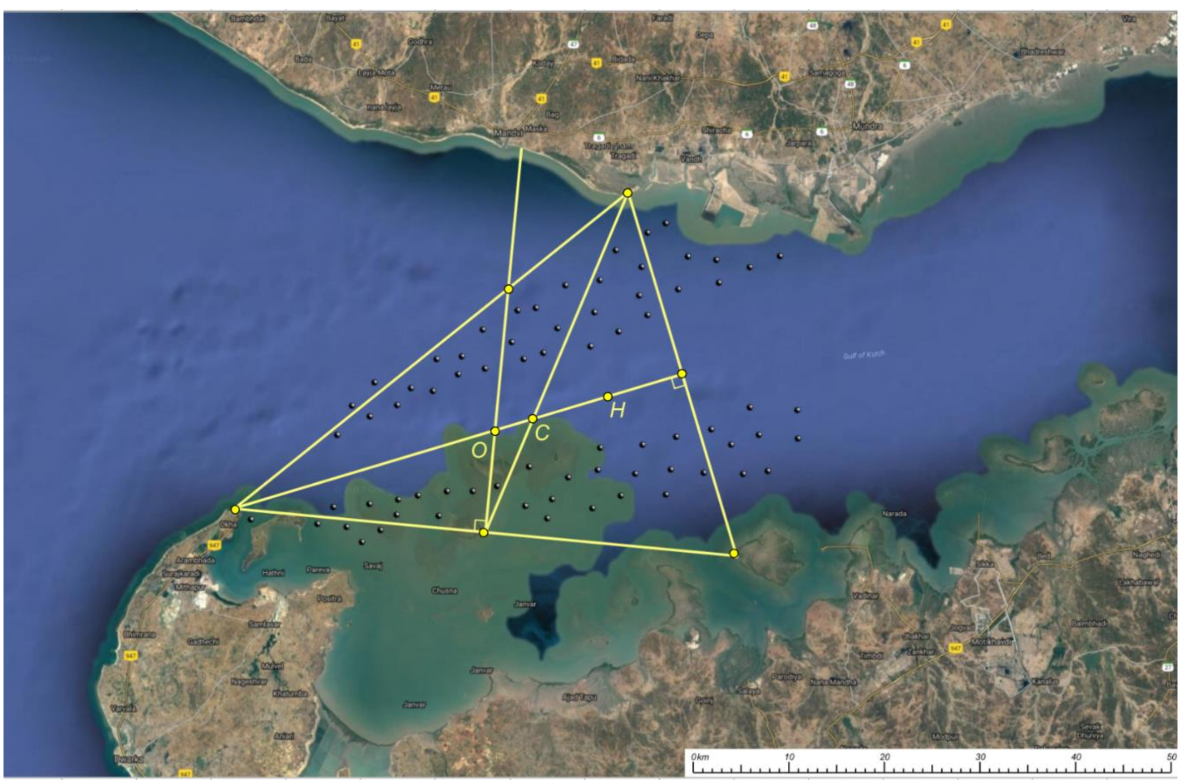
Solution

Let O be the circle center, P be the point farthest from O, and Q be the other point. Draw a circle around O through P. Let r be the radius of this circle, $r = |\overline{OP}|$. Q is somewhere inside this circle, which has area πr^2. By the tangent theorem, the vertex at P is acute. For the vertex at O to be acute, Q must be in the semicircle that P is on the center line of. For the vertex at Q to be acute, by Thales' diameter theorem, Q must be outside the circle of radius $\frac{r}{2}$ centered at M_{OP}. Thus, for the triangle to be acute, Q must be in a region of area $\frac{\pi r^2}{2} - \pi\left(\frac{r}{2}\right)^2$. Divide by πr^2 and simplify: The probability is 25%. ■

Strategic Defense Applications of Geometry

Scenario #1: Your roughly circular island has three fairly evenly spaced port cities connected by straight highways. There are many guns aimed seaward at the cities, but there is concern that the enemy might land troops on the coast and then bring guns to bear on the cities from the hills. The army can quickly move troops on paved roads, but not through the muddy interior. Because there are always traffic jams near the cities, the army has decided to build a fort on each of the three highways and connect them directly to each other with roads through the interior that are restricted to military vehicles. Where should they build?

Scenario #2: Suppose that India and Pakistan are at war. Guided missiles are not free, and the Indian Army has allocated them elsewhere, so you are tasked with defending the Gulf of Kutch with guns at Okha that you will supplement with two big mortars near Tragadigham and the round island north of Salaya. Knowing that the Euler segment coincides with the center line of an isosceles triangle, you position your mortars to be equidistant from Okha so, by the mediator theorem, they can fire simultaneously and have the same ballistics on any warship moving down the center line where naval mines have left open a corridor for commercial shipping. To time the firing, the ships will break two laser beams so, at constant velocity, the time between passing the circumcenter, O, and the medial point, C, is exactly half the time it will then take them to reach the orthocenter, H. All the while, the guns at Okha will be raking them from stern to stem!



Scenario #3: Suppose that you are a colonel in command of three army bases at distances of $10, 12$ and 16 klicks from one another. The general wishes to construct a munitions dump inside your triangle and to further defend it with three antiaircraft guns that form an equilateral triangle with the munitions dump at its center and your three bases on each of the three sides of the equilateral triangle. He insists that this equilateral triangle be as large as possible.

Queſtion CCCXCVI. *By Mr.* Tho. Moſs. L. D. 1735.

In the three ſides of an equi-angular field ſtand three trees, at the diſtances of 10, 12 and 16 chains from one another: to find the content of the field, it being the greateſt the data will admit of.

This is the Moss problem from page 235; did you get it? This first appeared in the *Ladies Diary*, though it was posed by a gentleman, Thomas Moss. He did not think of the military application, probably because antiaircraft guns did not exist in 1735. But Daniel Bernoulli's *Hydrodynamica* was published only three years later, which was the axiomatic theory that the Wright Brothers applied 165 years later, in 1903. As so often happens, axiomatic theory precedes its application by one or two centuries, at which time data is finally generated to illustrate the abstract theory.

If it had not been for David Hume, the English could have met General Washington with biplanes in 1776. I discuss how David Hume held back science in my earlier book (Aguilar, pp. 98-99).

> *"When I see, for instance, a billiard ball moving in a straight line toward another,... may I not conceive that a hundred different events might as well follow from that cause?... All these suppositions are consistent and conceivable. Why then should we give preference to one which is no more consistent or conceivable than the rest (David Hume, p. 50)?"*
>
> *However, the term "consistent" is only meaningful in reference to a specific set of axioms, in this case those of Newtonian Mechanics. Only one of the hundreds of different events mentioned in the above quotation conserve both momentum and energy, so, in reference to these two axioms, all the other events are inconsistent. The purpose of having a theory at all is that one does not have to apply experience to every event but need only apply it once when deciding on one's axioms, in this case that momentum and energy are both conserved.*

Applying experience to every event is perpetuated today by people like Jean-Philippe Bouchaud, who engage in blind guesswork while desperately praying that "statistical regularities should emerge." The statistics that logical positivists boast so proudly of are nothing but a history of their failures. And this is a history of failures that dragged on for centuries *after* the axiomatic theory was already in place that could have solved the problem that their guesswork was blindly stabbing at. The same thing happened to Leonhard Euler, whose 1745 axiomatic theory of ballistics lay dormant until WWI, when artillerists finally stopped stumbling around stooped over and staring at the dirt while filling fat binders with vast and incoherent data on cannonball marks.

There is no such thing as statistical physics!!! All real physics – 100% of it – is based on axioms.

But now, let us get busy solving the Moss problem. Tomorrow, three flatbed trucks will arrive with antiaircraft guns, followed by a train of cement trucks turning wet cement. The drivers need to know where to position the guns and where to pour concrete for the munitions dump. Hume and his acolytes (*e.g.,* Bouchaud) are welcome to stumble around for centuries taking random measurements while hoping beyond hope that "statistical regularities should emerge." But we need a solution *today*; only the axiomatic method can deliver provably true results this fast!

> ***Solution***
> Build equilateral triangles on the sides of the given triangle and connect their apexes to the given triangle's opposite vertices. These segments concur at the Torricelli point. At each given vertex, draw perpendiculars to the Torricelli segments; they intersect at the vertices of the largest equilateral triangle that has the given vertices on its sides. ■

Let $\overline{JKL}$ be not too obtuse as, indeed, a $10 : 12 : 16$ triangle is; the Torricelli point, U, is inside $\overline{JKL}$. $\overline{EFG}$ is the constructed triangle with J, K, L on $\overline{FG}, \overline{GE}, \overline{EF}$, respectively. We must prove: (1) $\overline{EFG}$ is equilateral; and (2) $\overline{EFG}$ is the largest equilateral triangle that has J, K, L on its sides.

> ***Proof of Part 1***
> By the Torricelli angles theorem, $\angle JUK = \angle KUL = \angle LUJ = 2\varphi$. By the right cyclic theorem applied to $\overline{ELUK}$, $\overline{FJUL}$ and $\overline{GKUJ}$, $\angle E = \angle F = \angle G = \varphi$. ■

> ***Proof of Part 2***
> By the Viviani equilateral theorem, $\overline{UJ} + \overline{UK} + \overline{UL}$ is the altitude of $\overline{EFG}$. Suppose there is another equilateral triangle that has J, K, L on its sides. J, K, L cannot be the feet of perpendiculars dropped onto this triangle's sides from U, so let us call these feet J', K', L'. $\angle UJ'J = \rho$, so $\overline{UJ'J}$ is right and $\overline{UJ'} < \overline{UJ}$ by the greater side theorem. Analogously, $\overline{UK'} < \overline{UK}$ and $\overline{UL'} < \overline{UL}$. By the Viviani equilateral theorem, the altitude of this other equilateral triangle is $\overline{UJ'} + \overline{UK'} + \overline{UL'} < \overline{UJ} + \overline{UK} + \overline{UL}$. ■

If $\angle L > 2\varphi$, then we must accept the "Torricelli point" being outside $\overline{JKL}$ and write an angles theorem for it: U is the "Torricelli point" of $\overline{JKL}$ if and only if $\angle KUL = \angle LUJ = \varphi$. The solution to the Moss problem is the same; the proof – left as an exercise – uses the Viviani sum theorem.

Note to Philosophers

The most fundamental question of philosophy is what it means to be human. We used to think that tool making was the defining characteristic until Jane Goodall observed chimpanzees dipping for insects with purpose-made twigs. Stone is harder to work, but that does not make a man. I argue that the defining characteristic is abstract reasoning using symbols to represent things. If I hide a child's toy and then show her a drawing of the floor plan of our house and I say, "This is our house, here is your room, here is the kitchen and X marks the spot where your toy is hidden," she will look at the drawing and then go get her toy. This would not work with our dog. I could draw maps on the floor every day and he would die of old age without ever showing a glimmer of understanding.[105] It is geometry, not engineering, that distinguishes us from the animals.

Immanuel Kant spoke of Euclidean geometry and is widely derided today by people who feel that the work of Lobachevski and Bolyai consign *Critique of Pure Reason* to the rubbish bin. But at the time of publication, 1781, there was only one geometry; the work of Lobachevski and Bolyai came 50 years later and Riemann 25 years after that. What matters is not that Kant used the term Euclidean as though it were synonymous with geometry but that he cited only theorems before Book I, Proposition 29 of *The Elements*, the first theorem to use Euclid's fifth postulate.[106]

What Kant really meant is that you do not have to explain to children that a segment being straight implies that it is the shortest path between two points, or that the points on the shortest path are between the two points. Basic concepts like what points and lines are and what it means for a line to be straight or for a point to be between two other points do not require explanation; the geometry teacher is just assigning names to concepts that the child already understands. This is the same point that the Epicureans made when they scoffed at the triangle inequality theorem for being evident even to an ass, who knows what the shortest path to a bale of hay is.[107]

Some geometers define a segment as two points and all those between them. But the shortest path and the points between are the same thing. A point would not be thought of as between two others if it were not on the shortest path from one to the other. Defining a segment as the shortest path between two points is more useful because there are lots of problems in geometry about minimizing lengths; *e.g.,* problems 3.2 to 3.4 and the problems of Torricelli and Fagnano. Our axioms are chosen not just to be intuitive to a child, but also to be productive for the adults.

[105] Of course, the dog has instincts that the child lacks. When I blew a dying-rabbit call inside my house, my dog, who had never seen a rabbit, went nuts looking for it while the little girl just covered her ears. So, while she scoffs at him for not reading maps, he scoffs at her for not helping pull the couch back to get at the dying rabbit.
[106] Bolyai coined the term neutral geometry to mean what is common to Euclidean and hyperbolic geometry.
[107] Toss bread onto a duck pond and watch the ducks define the geometric meaning of a segment with their wakes.

I will now quote at length from my earlier book, whose first three chapters were about the philosophical foundations of economics, though epistemology is relevant to any science that has real-world applications. I began the epistemology chapter with a quotation of Immanuel Kant:

> *Reason, holding in one hand its principles… and in the other hand the experiment which it has devised according to those principles, must approach nature in order to be taught by it; but not in the character of a pupil who agrees to everything the master likes, but as an appointed judge who compels the witnesses to answer the questions which he himself proposes. – Kant*

I began (p. 18) with the observation that "there are three senses of the term 'truth:'

1. *The observation that a phenomenon exists which conforms to a certain definition, or the observation that no phenomena exist which conform to that definition;*
2. *The applicability of a theory to a situation so that the relations or the characteristics to which those phenomena conform can be predicted by their conformance to the other, or the contingency of that theory's application on the conformance of phenomena not yet observed; and*
3. *The ability of a theory to apply without contradiction to some situation, or the impossibility of that theory to apply to any situation because its every alternative contains a contradiction.*

When I wrote this in 1999, I thought everybody understood how logical deduction got its name, but apparently this is not the case, so I will explain. Willard Quine (p. 69) gives an example:

$$-\{p \longrightarrow \bar{s}q. \longrightarrow -(sq \longrightarrow p): -[-(rp) - (p \longrightarrow \bar{s})]\}$$

Through various machinations (pp. 69–71), he converts this into alternational normal schemata:

$$\bar{p}\,\bar{s} \lor \bar{p}\,\bar{q} \lor \bar{p}\,p \lor \bar{s}\,q\,\bar{s} \lor \bar{s}\,q\,\bar{q} \lor \bar{s}\,q\,p \lor \bar{r}\,p\,s \lor \bar{p}\,p\,s$$

Quine writes, "We can quickly shorten this result by deleting the patently inconsistent clauses '$\bar{p}\,p$', '$\bar{s}\,q\,\bar{q}$', and '$\bar{p}\,p\,s$'. We then have:

$$\bar{p}\,\bar{s} \lor \bar{p}\,\bar{q} \lor \bar{s}\,q\,\bar{s} \lor \bar{s}\,q\,p \lor \bar{r}\,p\,s$$

Such deletion is a case of the procedure explained in Chapter 6: each of the patently inconsistent clauses may be thought of as supplanted by '⊥', which afterwards drops by resolution."[108]

This is how logical deduction gets its name; the logician is deleting the inconsistent clauses.

[108] In logic, ⊥ means inconsistent; this differs from geometry, where the symbol means perpendicular.

Much of the criticism of my book in the ensuing twenty years seems based on people – including professional philosophers – thinking that "deduction" is just a name. Deduction establishes third-sense truth by deleting the inconsistent clauses; if they all get deducted, then the statement is untrue in the third sense; that is, it is inconsistent. Quine (pp. 74–75) presses simplification further by using the seven forms of simplification from his Chapter 9. This deduction results in:

$$\overline{p}\,\overline{s} \vee \overline{p}\,\overline{q} \vee \overline{s}\,q \vee \overline{r}\,p\,s$$

Quine continues, "The initial clause is in fact redundant; it is equivalent to:

$$\overline{p}\,\overline{q} \vee \overline{s}\,q \vee \overline{r}\,p\,s$$

"There is a quick way of testing any clause of an alternational normal schema to see if it can be thus dropped as redundant. The law (vii) of Chapter 9 tells us how: just check, by fell swoop, whether the clause implies the rest of the schema...

"Two good ways are now before us for simplifying alternational normal schemata. We can test a clause for redundancy, and we can test a literal for redundancy, in each case by fell swoop. An alternational normal schema can, however, resist both redundancy tests and still admit of simplification in more devious ways. An example is:

$$p\,\overline{q} \vee \overline{p}\,q \vee q\,\overline{r} \vee \overline{q}\,r$$

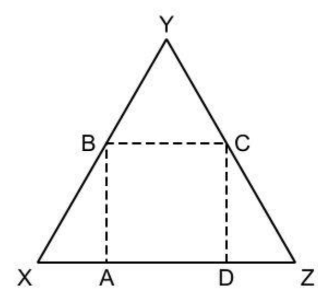
By twelve fell swoops the reader can test each clause and each literal for redundancy and draw a blank every time. Yet it has a simpler equivalent, $p\,\overline{q} \vee \overline{p}\,r \vee q\,\overline{r}$."

It absurd that the logical positivists (Ayer, *et. al*.) illustrate deduction with ridiculous examples like "all bachelors are unmarried men." Ayer sneers at the work of mathematicians as being trivial, giving the example $91 \times 79 = 7{,}189$ as something that we are capable of, unlike the common man, who is only capable of calculating $7 + 5 = 12$. Geometer, does Ayer's description of your accomplishments have you just busting out with pride? Not. Ayer should never have used the word "logic" in the title of his book. Logical positivists know nothing about logic!

But now, having shown that analytic *a priori* knowledge – the results of logical deduction – is just a teensy bit more difficult to come by than the drivel that logical positivists accuse it of being, let us turn our attention to something that they know even less about: synthetic *a priori* knowledge!

Axiomatic Theory of Economics has a section (pp. 43–48) titled, *The possibility of synthetic* a priori *knowledge*. But economists – who are all logical positivists – just sneered and shrieked accusations of insanity (autism) at me. Until Steve Keen *et. al.* started calling me "autistic," I had never heard the word before; I had to look it up in the dictionary. "Autistic" – that, I am not.[109]

> *While there are a finite number of theories implied by a given one, there are an infinite number of theories that could have implied a given theory. These theories are found by adding definitions (which are always available) to an alternative of the given theory. If, out of this infinity of theories, the analysis of one of them yields an alternative that contains only the characteristics of the given theory (which is implied by the one under analysis) and yet imply a relation that is not contained in that given theory, this relation is synthetic* a priori *knowledge.*
>
> *The use of additional definitions which are then deducted after a solution has been found is often forgotten, leading people to believe that synthetic* a priori *knowledge is impossible and that all understanding is analytic. That synthesis is a passing event which leaves no mark on its creation and that all declarative sentences are analyzable from discursive postulates has led many linguists to take this stand. As linguists deal with theories whose creation has been forgotten and which have turned into statements that could have easily been handed down from a mountain as synthesized, it is not surprising that they should regard them as analytic knowledge. They need only ask "What do the words mean in this configuration?" and they know the meaning of the theory. They forget that at one time the theory was unknown but a simpler one was known without certain relations.[110] Then people noticed that, whenever they used the theory, the phenomena that conformed to it had those relations, but they were hesitant to risk anything on the assumption that future phenomena would have those relationships also, for they could not be sure that it was not a coincidence. Then someone found an anti-implication of the theory which, when analyzed, yielded those relations as synthetic a priori knowledge... An illustration of synthetic a priori knowledge will now be given:*
>
> *If definition p is of a given equilateral triangle XYZ and definition q is of a square, what additional definition(s) r must be added to pq so that the analysis of pqr leaves only the characteristics of p and q (there are no superfluous lines) but with the square given definite size, $ABCD$, so that it fits inside triangle XYZ , as shown to the right?*



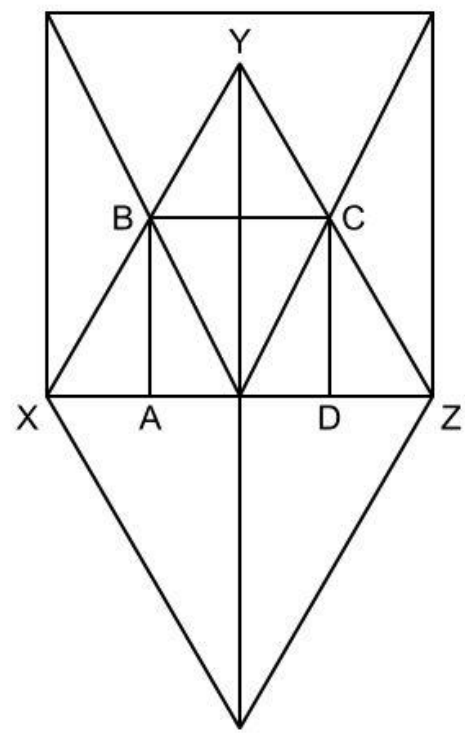
This is P. 5.7, an application of homothecy, but I solved it when I was 14 and when I had not heard of homothecy. So, even if you are not a geometer, you should still try it before turning the page.

[109] Also, until Keen *et. al.* accused me of being a follower of Debreu, I had never before heard the name Debreu.

[110] For instance, it has long been known that the exterior angle of a triangle is greater than either remote interior angle. It looks like it is equal to their sum, but people could not be sure when taking measurements off a clay tablet. Then, some smart guy drew a line parallel to the base through the apex, proved the angle sum theorem, and erased his line. Descartes would later say this theorem is in the "essence" of a triangle; *i.e.* analyzed from its definition. No.

Neither the definition of p (three equal sides of definite length) nor the definition of q ([a square[111] with] four equal sides of indeterminate length) contains any information about the positions of B or C. Additional characteristics are needed (lines have to be drawn) to find these lengths. But these additional characteristics have to be deducted again for the new theory to keep the same extension. It is easy to delude oneself in this sort of exercise by assuming that the additional definitions needed are contained in the "full" meaning of the given definitions, even though they are not strictly expressed in them. It is the mark of an immature science to rely on such assumed meaning; only when a science defines its terms in the strictest way possible can it truly be called a science.

The figure to the right illustrates the solution. The first additional definition needed is of another equilateral triangle with XZ as one side. Next, connect the opposite vertices so that the line crosses XZ at its midpoint. Then, construct a square with XZ as one side (additional triangles must be constructed to prove the sides perpendicular to XZ, but they are not shown). Finally, draw lines from the midpoint of XZ to the upper corners of the new square. The points where they cross XY and YZ are the definitive positions of B and C. Connect B and C and drop perpendiculars from them to line XZ to find points A and D of the square inside the equilateral triangle. This figure, pqr, is an anti-implication of pq, as it can imply pq by deducting the additional lines just added. They are not needed for phenomena to conform to the square inside the triangle. After their deduction, however, the relation of B and C to lines XY and YZ, which was not known before, is still there. This relation is synthetic a priori *knowledge.*



A more algebraic example is the integration of $\frac{1}{\ln x}$. The first three steps establish the needed anti-implication.

$$\int \frac{dx}{\ln x} = \int \frac{1}{x\ln x} + \frac{x}{x\ln x} - \frac{1}{x\ln x}\, dx = \int \frac{1}{x\ln x} + \frac{e^{\ln x}-1}{x\ln x}\, dx = \int \frac{1}{x\ln x} + \frac{1}{x\ln x}\sum_{n=1}^{\infty}\frac{\ln^n(x)}{n!}\, dx$$

$$= \int \frac{dx}{x\ln x} + \int \sum_{n=1}^{\infty}\frac{\ln^{n-1}(x)}{xn!}\, dx = \int \frac{du}{u} + \int \sum_{n=1}^{\infty}\frac{u^{n-1}}{n!}\, du = \ln(\ln x) + \sum_{n=1}^{\infty}\frac{\ln^n(x)}{nn!} + c$$

Philosophers are invited to read the foundations, even if they cannot follow the later economics.

[111] I neglected to say "square;" also, I did not use overbars and used "line" loosely. I am a better writer now! The introduction is here: www.researchgate.net/publication/349861553_Foundations_Axiomatic_Economics-Victor_Aguilar

Note to Computer Programmers

Common Core geometry is sometimes just straight-out wrong; *e.g.,* turning everything into real numbers, including the lengths of segments and the measures of angles, and then doing a bunch of algebra that includes adding them together. But mostly it is just strange. Why, for instance, are transformations the so-called "spine" of *Common Core* geometry? Defining geometry as "properties of geometric figures that are not changed by motion" makes sense only to those who know linear algebra; without vectors, motion is nowhere defined. Vague references to advanced math that neither the students nor their teacher have had is what killed *New Math* in the 1970s.

So why are transformations the spine of *Common Core* geometry? Because Bill Gates spent over $200M in 2008 alone and just steam-rolled mathematicians that actually understand geometry but had zero funding. Why did he spend this money, and why is defining geometry as invariant properties of rigid figures under motion the only thing that even remotely resembles an ideology among *Common Core* shills? Gates may not be able to construct an equilateral triangle to save his soul, but he does know one thing about geometry: Pencil lead is impregnated in the paper. Geometric figures cannot be moved anywhere; they can be redrawn elsewhere on the paper, but that is all. Defining geometry in terms of motion is just another way of saying that it can only be done on a school computer running educational software purchased from you-know-who.

If lines are two pixels wide and a computer screen is 1366×768 pixels, a circle is typically no more than 200 units in diameter, about 600 units around. To rotate a geometric figure $60°$ around a circle is to redraw it about 100 times. Thus, there is not a single problem in plane geometry that a linear search of a few hundred possible solutions cannot produce a numerical answer to, and a multi-colored illustration of. Gates eliminated solid geometry because the solution space is too big. A century ago, George Wentworth taught plane *and* solid geometry – but not Gates! So why am I writing about plane geometry? Because the study of logic has virtue beyond just the drawing of pretty illustrations. When Wentworth was the American geometry master, it was understood that the primary responsibility of schools was to teach students to employ cool logic even in the face of adversity. Now schools shill for software moguls. Deducing geometry from postulates as I do is the high road to logic. Educational software does not do this.

When I was in high school thirty years ago, contractors and retailers refused to hire high-school dropouts. Today, most male dropouts have a construction job waiting for them, and the girls a retail job. These jobs are available to dropouts because employers are aware that a high-school diploma is worthless. A diploma is no assurance that an apprentice framer can square a wall or that a cashier can make change. So, please do not criticize my pencil-and-paper methods! Programmers use logic too. If squaring a wall or making change is hard, what is recursion like?

Geometry Informs the Numerical Analysis of Error in Computations

Recall that to find the touching point requires using loci that intersect at an angle closer to perpendicular than the angle made by the tangent and the circle. That Thales devised construction 4.4 to accomplish this is a large part of why he is still famous two millennia later.

Recall from Algebra I the basic technique for finding the x and y where two lines intersect.

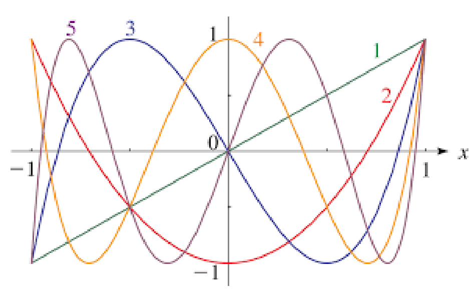
Unique Solution: $x = 2, y = 3$	Uniqueness Fails	Existence Fails
$y = x + 1$	$y = x + 1$	$y = x + 1$
$y = 2x - 1$	$2y = 2x + 2$	$y = x + 2$

In this context, locus means all the points that satisfy an equation; that is, the graph of the line defined by the equation. So, to solve two linear equations in two unknowns geometrically means to observe where their graphs (loci) intersect; to solve them algebraically means to perform algebra operations on $y = x + 1$ and $y = 2x - 1$ until they turn into $x = 2$ and $y = 3$.

But what about $y = mx + b$ and $y = (m + \varepsilon)x + c$? If ε is close to zero, then the two lines are so close to parallel that it is difficult to see where they intersect; both existence and uniqueness are on the verge of failure. As ε approach zero, the solution approaches guesswork. For two linear equations in two unknowns, little can be done to improve accuracy, but for multiple linear equations in as many unknowns, much can be accomplished by rearranging them with a technique called scaled partial pivoting. This is college-level linear algebra.

Polynomial regression is a statistical technique for fitting an n^th degree polynomial to a set of data points such that $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ interpolates the experimental data. The problem is that x^n and x^{n-1} are almost the same function – graph them for x^6 and x^5 to see that they are almost on top of each other from zero to one – so this equation is not much different than $y = (a_n + a_{n-1})x^{n-1} + \cdots + a_1 x + a_0$. Increasing the degree of the polynomial from 5 to 6 was difficult but resulted in a negligible increase in accuracy. The numerical analysis solution is Chebyshev polynomials, the first few (T_6 is not in the image) being shown below:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$



Instead of finding the coefficients for the formula $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, we find the coefficients for the formula $y = c_n T_n(x) + c_{n-1} T_{n-1}(x) + \cdots + c_1 T_1(x) + c_0$. There is an easy way to evaluate this formula without having to raise x to any integer powers:

LET $w_{n+2} = 0$ AND $w_{n+1} = 0$
FOR $k = n$ TO 0 STEP -1
    $w_k = c_k + 2x w_{k+1} - w_{k+2}$
NEXT k
$y = w_0 - x w_1$

It is beyond the scope of this book to prove that this algorithm works or to show how Chebyshev came up with his polynomials, but I will say this: A set of functions that, within some range, have widely varying slopes instead of being like x^5 and x^6 are near zero, are called orthogonal. Given a sequence of functions like $1,\ x,\ x^2, \cdots$, there is a procedure called the Gram-Schmidt process that creates a sequence of functions that is orthogonal. In my 1999 book, *Axiomatic Theory of Economics* (p. 132), I had to approximate a function whose evaluation requires numerical integration and is thus time consuming. Also, it increases at roughly the acceleration rate of e^{x^2} and thus quickly outpaces any polynomial. I solved this problem by applying the Gram-Schmidt process to the sequence of functions $e^1,\ e^x, e^{x^2}, \cdots$.

Suppose that the ABC Chemical Company is manufacturing a product on a special-order basis for customers who specify a certain characteristic, y. ABC knows that this characteristic in the finished product is positively related to how much of a certain chemical, x, is added to the mixture before stirring it in a ball mill for a week and then baking it. But this is all they know and so ABC has hired you, young mathematician, to solve this problem for them. All that ABC can tell you is that, after four expensive and time-consuming experiments, they have these (x, y) pairs: $\left(0,\ \frac{2}{3}\right),\ \left(\frac{1}{3},\ \frac{25}{36}\right),\ \left(\frac{2}{3},\ \frac{59}{54}\right),\ \left(1,\ \frac{25}{12}\right)$. Ask your linear algebra teacher for help!

School problems with only four data points do not really need advanced mathematics; basic polynomial regression problems using $1,\ x,\ x^2$ and x^3 can be solved on a 32-bit computer without solving for $T_0,\ T_1,\ T_2$ and T_3. But real-life problems have more than four data points.

In the above problem, the even spacing of the x values indicate that these experiments were carried out for the express purpose of interpolation. But it is more likely that you will be presented with the results of random guesses made in the past to satisfy actual customer orders. For this, use a polynomial of lower degree than there are data points to get a smooth function that slightly misses them rather than a wildly varying function that hits them exactly.

A Reply to the Enemies of Deductive Logic

Grandparents proudly following their young grasshopper's progress in *Geometry–Do* (On obtaining a black belt, it is appropriate to be presented with a motorcycle. *Hint! Hint!*) and who sat at Euclid's feet may want to know where we are relative to *The Elements*. The last pages of yellow belt, about half of orange belt, three-quarters of green belt and all of red belt are beyond *The Elements*.[112] This puts the lie to critics who wish geometry to be just a review of algebra, such as Agostino Prástaro, who sneers at us for believing that mathematics ends with Euclid:

> *I can understand that for mathematicians of your ilk, namely fans of intrinsic geometry, mathematics ends with Euclid. But unfortunately (for you) this is not a universally shared opinion. On the other hand, it is a questionable opinion that a serious program in intrinsic geometry could be suitable to introduce high school students in geometry… By conclusion, in my opinion in order for geometry to be more attractive for high school students, it should be suitable to introduce it as an algebra application.*

George Birkhoff's axioms are called metric because they associate every length and angle with a real number; those of Euclid, David Hilbert and I are called intrinsic because they do not. Professor Prástaro[113] (University of Rome) is wrong about intrinsic geometry ending with Euclid. Great advances have been made in the intervening 2300 years, right up to the present; Franzsen published at a time when current high-school students were in elementary school.

But the most telling part of this quotation is the last line, where Professor Prástaro insists that geometry should be introduced as an algebra application. In the United States this is already mandated by *Common Core*. Sophomore geometry is a review of freshman algebra. The triangles serve no purpose beyond setting up Algebra I problems. How well does this work?

Orange Belt Exit Exam

Given two circles, a line and a length, construct a line parallel to the given one so it cuts the two given circles the given length apart. How many possible solutions are there?

If we follow Prof. Prástaro's advice to "introduce it as an algebra application," we must first come to grips with the fact that God, in his Wisdom, did not imprint a Cartesian coordinate grid on the surface of the Earth. It is only in *Common Core* textbooks that every point comes pre-labeled with its coordinates. So, let us take the given line as the x-axis and center a circle on the y-axis.

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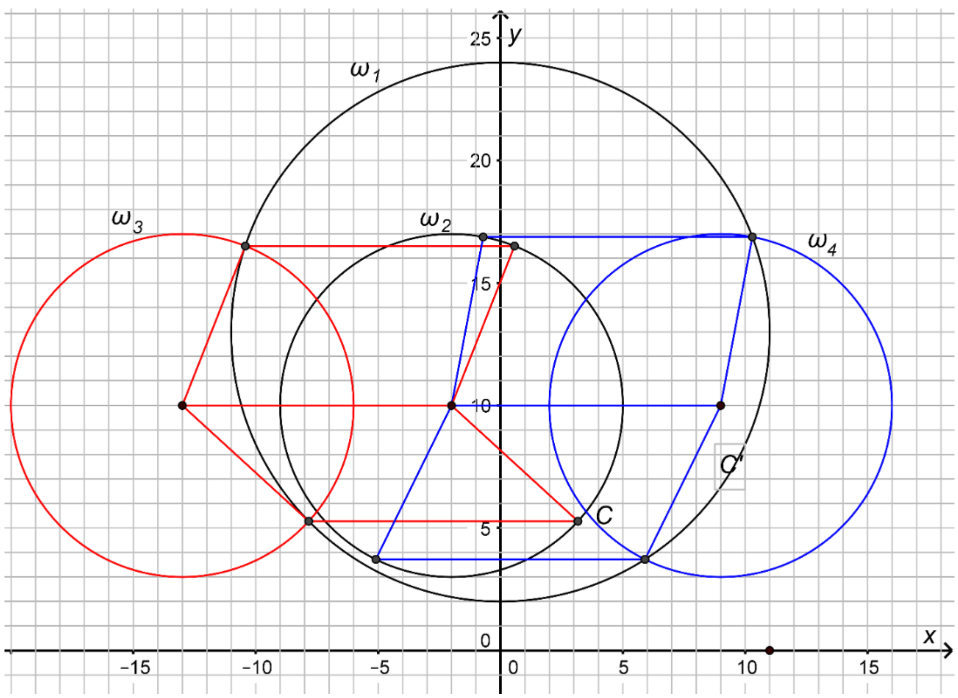
[112] Euclid proved quadrature theorems as soon as he could, which scattered them. I make all of quadrature blue belt, but bring easy theorems forward to an orange-belt appendix so first-year students know all of Euclid's Book I.

[113] A critique of Prástaro's work: davidlowryduda.com/reviewing-goldbach/#more-706

Two circles, ω_1 and ω_2, are defined by $x^2+(y-13)^2=11^2$ and $(x+2)^2+(y-10)^2=7^2$, respectively, and the given length of 11 units is laid off on the x-axis. We hope to find y_0 such that the intersection of the line $y=y_0$ and the circle $x^2+(y-13)^2=11^2$ is found at (x_1,y_1) and the intersection of the line $y=y_0$ and the circle $(x+2)^2+(y-10)^2=7^2$ is found at (x_2,y_2) and the equation $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=11$ is satisfied. This is a *lot* of algebra!

Geometry–Do students will immediately recognize that, if we are looking for a segment that is both parallel to and equal in length to a given segment, then we are going to construct a parallelogram. Construct circles ω_3 and ω_4 congruent to ω_2 but 11 units to the left and the right. If they intersect ω_1 – *No guarantees of intersection!* – then the horizontal distance from the intersections to ω_2 is 11 units. In this case, ω_3 and ω_4 each intersect ω_1 twice, so four solutions.



Common Core geometry standards[114] denigrate deductive logic, "in college some students develop Euclidean and other geometries carefully from a small set of axioms," emphasis on *some*, implying that high-school teachers are forbidden from developing geometry from a small set of axioms, as we do. Russian 12-year-olds laugh when they read, "During high school, [American 10th grade] students begin to formalize their geometry." By 10th grade the Russians will be studying Ceva's theorem. And hate for deductive logic is not just in America; it got its start in France. Their core curriculum states, "teut expos'e de logique formalle est exclu" [any formal logic is excluded], and Jean Dieudonné coined the slogan, "A bas Euclide! Mort aux triangles!"

[114] It's not just a suggestion, it's the law! www.corestandards.org/Math/Content/HSG/introduction

David T. Conley[115] (p. 3) boasts of *Common Core*:

> *International comparisons also helped ensure the standards were set at a high level. For example, the Third International Mathematics and Science Study (TIMSS) yielded detailed profiles of how numerous other countries teach math, which assisted in identifying the most effective sequencing of mathematics topics.*

TIMSS may have talked to the Italians, but they certainly did not talk to the Russians! Even without the ability to read Russian, it is clear from the title of Pogorelov's textbook[116] that this is for 7th grade students. A Russian student that has completed the first six chapters of Pogorelov – 7th grade is five chapters and the sixth is completed a month into 8th grade – can pass David Conley's so-called "internationally benchmarked" standards for American high-school graduates. *Common Core* is all about memorization and Conley's exam requires students to memorize but not prove a few theorems beyond the first six chapters of Pogorelov.

Common Core is a charade. David Conley is unknown in mathematics but, by his writing style, it is clear that he is much more closely associated with Madison Avenue than with any university. If he did attend college, I am certain that he never darkened the door of their mathematics department.[117] He refers to himself as a "Professor of Leadership," whatever that is. David Conley has posted 70 papers at Research Gate and received less than 700 downloads, total. By comparison, my manuscript[118] was downloaded over 700 times in its first five months at Research Gate, from January to June 2016. This is a startling statistic considering that – until I defeat *Common Core* and am legally allowed to publish – I am doing this for free. The taxpayers purchased every one of David Conley's 70 papers with government grants averaging about a quarter of a million dollars per paper – a lot of money for papers downloaded less than ten times apiece. It is clear that nobody but his socialist masters actually care what David Conley writes. Equally obvious is that – despite using "college ready" as their favorite buzzword – the socialists are turning *Common Core* students into worker bees for their government-owned factories.[119] Socialists want laborers who are literate enough to read the operating manual for the machines they are to toil at, but not literate enough to read books. Certainly not books about logic!

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[115] A blizzard of buzzwords intended to both baffle and befuddle you! www.inflexion.org/download/36952

[116] This is in Russian, but the title is readable: www.axiomaticeconomics.com/Pogorelov_geometry_7-9.pdf

[117] I found David Conley's *curricula vitae*; he has a BA in social sciences, an MA in multiculturalism and a PhD in school administration. There is no evidence that he has ever taken a college-level mathematics class in his life. The only actual teaching experience of this pasty-faced multiculturalist is an Ethnic Heritage Program in Jefferson County Public Schools from 1978 to 1982. The rest of his life has been spent dictating to people who actually majored in the academic subject they teach exactly what material they are and are not allowed to teach.

[118] www.researchgate.net/publication/291333791_Volume_One_Geometry_without_Multiplication

[119] Strictly speaking, it is called fascism, not socialism, when big companies like McGraw Hill or Pearson bribe their way into a monopoly. And the big factories are owned by stockholders, though effectively they are the government.

On When to Use Algebra Instead of Geometry

In the *Comparison with Common Core Geometry* at the end of the white belt chapter, I mock *Common Core* proponents for turning every problem into an algebra problem.

> ***Teachers!*** *If you have read this far hoping for advice on how to get your #%$^@ students through the* Common Core *standardized exam, here it is: Ask for the perimeter of a triangle with vertices* $(-2,3), (-4,-4), (-7,-1)$ *and make it a race. The easy way is to lay the three sides end-to-end on a line. Taking the sum of three applications of the algebraic distance formula is the hard way.*
>
> $$\sqrt{(-2-(-7))^2+(3-(-1))^2}+\sqrt{(-2-(-4))^2+(3-(-4))^2}+\sqrt{(-7-(-4))^2+(-1-(-4))^2}\approx 17.9$$

So true! I have interviewed American teachers and, when I point out that they do not require their students to purchase a compass, they have gotten defensive, sifted through their desk drawer until they found one, and insisted that they had once demonstrated bisecting an angle.

A geometric construction is the best way to find the perimeter of a triangle drawn on graph paper. Even in the absence of graph paper, it is the easiest way to solve problem 1.24.

Problem 1.25 *Given a triangle with base* 14 *mm and legs* 13 *mm and* 15 *mm, what is the height?*

The preceding section, *A Reply to the Enemies of Deductive Logic*, presents yet another example of a problem that is far easier to solve with geometry than algebra. But I am not an implacable enemy of algebra; I have taught algebra and I consider myself at least competent in the subject.

The ancients struggled with some problems that we now know cannot be solved with geometry, the most famous being angle trisection. With two 30-meter strings a geometer can bisect an angle to as much accuracy as a surveyor with hundreds of dollars of optical equipment can trisect it. A square with the same area as a unit circle has sides of length $\sqrt{\pi}$, which is an ancient problem that motivated the algebraic approximation of π. Legend has it that the god Apollo punished the people of Delos with a plague. The oracle of Delphi informed them that they must double the volume of their cubic Apollo altar. A segment $\sqrt[3]{2}$ long cannot be constructed geometrically – cube roots require algebra – and Apollo would sooner see them all die than to accept 1.26.

The mirror problem is easy (yellow belt), but a closely related problem is impossible.

Mirror Problem

Given a light and an observer, find the point on a mirror to shine the light at the observer.

If a laser is shined at an arc, it reflects in the same way as it would off a flat mirror that is tangent to the arc at the aiming point. Initially, I wanted to end the yellow-belt chapter with a fun adventure story about a geometry student who is abducted by aliens and forced to fight other captive humans with lethal laser guns. The arena is in a tank farm on the far side of the moon, where the aliens are stockpiling fuel and oxygen for their planned attack on Earth. But, instead of shooting at the other humans, the boy uses geometry to find the aiming points on the tanks that he needs to hit the aliens, who are observing from behind a laser-proof screen. Having killed the aliens, he then leads the other humans to commandeer an alien battleship, bomb the alien military base and save the world! Great fun, but – *Ahem!* – our story is impossible.

The impossible part of this story is not that there is an alien military base on the far side of the moon – there may well be – but that the aiming point on a cylinder cannot be constructed with a compass and straightedge. For a cue ball and an object ball inside a round billiard table – rather than outside a steel tank – this is the Alhazen Billiard Problem first proposed by Ptolemy in 150 but made famous by al-Haytham *c.* 1020; it was proven impossible by P. M. Neumann in 1998.

It is possible to solve this exactly with algebra, but this requires first setting up a quartic (fourth-degree polynomial equation) and then solving it with Ferrari's formula. This is not high-school algebra; it is not undergraduate algebra either. This would be a challenging problem for a graduate student in the mathematics department – definitely not in the education department like your teacher – and one that is specializing in advanced algebra. I would do it numerically:

If the tank center is at the origin and of radius 20 m, suppose the gunner is at $(22, 24)$ and the alien is at $(20, 35)$ and behind a screen. If $(x_0,\ y_0)$ is a point on the tank, then $y_0 = \sqrt{400 - {x_0}^2}$. The line from the center of the tank through $(x_0,\ y_0)$ is $y = \frac{y_0}{x_0} x$ and so, for the angle of incidence to equal the angle of reflection, we must have $\operatorname{atan}\frac{y_0}{x_0} - \operatorname{atan}\frac{24-y_0}{22-x_0} = \operatorname{atan}\frac{35-y_0}{20-x_0} - \operatorname{atan}\frac{y_0}{x_0}$; the left and right sides are the complements of the angles of incidence and of reflection, respectively.

FOR $x = 8$ TO 16 STEP 2^{-12}

$y = \sqrt{400 - x^2}$

$z = \left|2\operatorname{atan}\frac{y}{x} - \operatorname{atan}\frac{24-y}{22-x} - \operatorname{atan}\frac{35-y}{20-x}\right|$

IF $z < 0.0001$ THEN PRINT x, y, z

NEXT x

The aiming point is approximately $(12.0271, 15.97964)$. Incrementing in an integer power of two reduces round-off error. Make the step size a 2^{16} part of the maximum, in this case 16. In QBasic, 2^{-12} is written 1 / 2 ^ 12, square root is SQR, absolute value is ABS and arctangent is ATN.

Pedagogic Instruction for North American Teachers

The first task of the American high-school geometry teacher is to disabuse students of the notion that geometry is just a boring review of Algebra I. (Nothing new here. *Blah!!!*) And the first step towards disabusing the students of this misguided notion is disabusing yourself of it. So, teacher, take the following exam and – only after you have tried your hardest – turn the page upside down to learn how a *Geometry–Do* practitioner would solve these problems. Use any method that you think will work. Be like the ancient Greeks and use a compass and straightedge; be like the modern Americans and use a scientific calculator; consult your horoscope; whatever gets it done.

1. **Yellow!** E is $(16, 16)$, F is $(2, 9)$, G is $(17, 6)$ and P is $(13, 9)$. Find $J \in \overrightarrow{FE}$ and $K \in \overrightarrow{FG}$ such that $\overline{PJK}$ is of minimal perimeter. Round the answer to the nearest hundredth.

2. **Yellow!** You are given a point $(-6, -5)$; a line with equation $2x - 11y = 109$; and a circle with equation $x^2 + y^2 = 25$. Find a point on the line such that the given point is exactly halfway between it and the circle. Round the answer to the nearest hundredth.

3. **Orange!** Three roads, $\overleftrightarrow{EF}$, $\overrightarrow{EG}$, $\overrightarrow{FG}$, would make a triangle, $\overline{EFG}$, with vertices E at $(-120, -110)$, F at $(90, -160)$ and G at $(0, 240)$, in meters. But, instead of making sharp turns at the vertices, $\overleftrightarrow{EF}$ will have exits to curve into $\overrightarrow{EG}$ and $\overrightarrow{FG}$ that are arcs of the ω_Z and ω_Y excircles. What is the distance between the exits to $\overrightarrow{EG}$ and $\overrightarrow{FG}$ on $\overleftrightarrow{EF}$?

4. **Green!** Given a triangle $\overline{EFG}$, let G' be the foot of the altitude from G and call this the origin. If E and G have coordinates $(-15, 0)$ and $(0, 112)$, respectively, and $\overrightarrow{GG'} \cap \omega$ is $(0, -36)$, with ω the circumcircle of $\overline{EFG}$, what are the coordinates of its orthocenter?

5. **Red!** The coordinates of the first Torricelli apexes are $(0, 0), (-26, 43)$ and $(20, 53)$. What are the coordinates of the vertices of their associated triangle?

Yellow: Plot P and draw $\angle EFG$. Reflect P across $\overrightarrow{FE}$ to P_{FE}. Reflect P across $\overrightarrow{FG}$ to P_{FG}. J and K are the intersection of $\overline{P_{FE}P_{FG}}$ with $\overrightarrow{FE}$ at $(9.92, 12.96)$ and with $\overrightarrow{FG}$ at $(11.52, 7.10)$, respectively.
Yellow: Plot the point and draw the circle on centimeter graph paper. Draw the line and then reflect it across the point and see where it intersects the circle. Draw lines through these intersections and the given point to see where it intersects the given line. Two solutions! $(-16.12, -12.84)$ and $(-7.15, -11.21)$.
Orange: By the excenter theorem corollary, it is $\overline{EG} + \overline{FG}$. Use your compass to lay them out on the horizontal; $\overline{EG} + \overline{FG} = 370 + 410 = 780$ meters.
Green: By the orthocenter and circumcircle theorem, reflect $(0, -36)$ around $\overleftrightarrow{EF}$ to $(0, 36)$. This is the orthocenter.
Red: The reverse Torricelli problem is a corollary of the equilateral sum theorem; $(-15, 26), (15, 26)$ and $(-6, 44)$.

Are you convinced that geometry is easier than algebra? Now let us consider pedagogy.

American teachers often dress up, like a history teacher dressing as an historical character. This is a practice that they apparently learned of in their five years of education classes (Can you say, *"Never lecture!"*) when they were not learning the content of their specialty. But I tell you:

1. Do not wear a black belt unless you are one. If and only if you pass the green belt entrance exam, you can wear a green belt; otherwise, you are orange belt. There is nothing wrong with an orange belt being the highest ranked geometer at a school that teaches shop students white- and yellow-belt geometry – somebody must be the leader.

2. There is nothing wrong with lecturing. The best way to convey information to people is to tell it to them. "Discovery math" is just another word for laziness. The teacher waits for a bright student to solve the problem and then she takes credit for the discovery.

At the beginning of the green-belt chapter I write:

> *We have now proven over a hundred theorems based on only our six geometric postulates. (The circle postulate is needed whenever we say, "construct an isosceles triangle.")*

Did you, teacher, ask yourself exactly when you had tacitly invoked the circle postulate? Here I quote from the white-belt chapter. [Edit: This was shortened, but I will quote it in full here.]

> *In the following construction, existence and uniqueness of $\overrightarrow{FE}$ and $\overrightarrow{FG}$ requires invoking the line postulate, though this goes unsaid. In the same way that, given E and F, we speak of $\overline{EF}$ without bothering to invoke the segment postulate, we now speak of $\overrightarrow{FE}$ or $\overleftrightarrow{FE}$ whenever $\overline{FE}$ has been defined. This practice is in keeping with our plan to avoid tedious proofs with mincing steps, but the student should never forget that Euclid's postulates are ever-present and needed.*
>
> ***Construction 1.1*** *Bisect an angle.*
>
> > ***Solution***
> > Given $\angle EFG$, take any point J on $\overrightarrow{FE}$. There exists a point K on $\overrightarrow{FG}$ such that $\overline{FJ} = \overline{FK}$. Construct an isosceles triangle with base $\overline{JK}$ and apex L on the other side of $\overleftrightarrow{JK}$ from F. By SSS, $\overline{JFL} \cong \overline{KFL}$, which holds the equality $\angle JFL = \angle KFL$. ■

We now carry out this construction again, this time presented in the traditional two columns:

$\angle EFG$	Given.
Take any point J on $\overrightarrow{FE}$.	Segment postulate implies $\overline{FE}$ exists. Line postulate implies $\overleftrightarrow{FE}$ exists.
There exists a point K on $\overrightarrow{FG}$ such that $\overline{FJ} = \overline{FK}$.	Segment postulate implies $\overline{FG}$ exists. Line postulate implies $\overleftrightarrow{FG}$ exists. Equal magnitudes are an equivalence relation and can be reproduced wherever needed.
Construct an isosceles triangle with base $\overline{JK}$ and apex L on the other side of $\overleftrightarrow{JK}$ from F.	Choose an arbitrary length. Circle postulate implies a circle with center J and this radius is fully defined. Circle postulate implies a circle with center K and this radius is fully defined. Each circle is the locus of points that satisfy one side of the triangle being of this length. Where these loci intersect are the points that satisfy the definition of an isosceles triangle. Find an intersection L on the other side of $\overleftrightarrow{JK}$ from F so $\overline{JL} = \overline{KL}$. If it does not exist, then try a longer radius until it does.
By SSS, $\overline{JFL} \cong \overline{KFL}$.	$\overline{JF} = \overline{FJ}, \overline{KF} = \overline{FK}$ by length definition $\overline{JF} = \overline{KF}$ by construction $\overline{FL} = \overline{FL}$ by reflexivity $\overline{LJ} = \overline{LK}$ by construction
By congruency, $\angle JFL = \angle KFL$.	Corresponding magnitudes of congruent triangles are equal.

We introduced the term locus when constructing an isosceles triangle. It also applies to the mediator and the angle bisector, which are the locus of points equidistant from the vertices and from the sides, respectively. Note “all” in the definition of locus; thus, the iff in these theorems. A student beginning geometry should know how to solve two linear equations in two unknowns when a unique solution exists, and what it means for either uniqueness or existence to fail.

Unique Solution: $x = 2, y = 3$	Uniqueness Fails	Existence Fails
$y = x + 1$	$y = x + 1$	$y = x + 1$
$y = 2x - 1$	$2y = 2x + 2$	$y = x + 2$

In this context, locus means all the points that satisfy an equation; that is, the graph of the line defined by the equation. So, to solve two linear equations in two unknowns geometrically means to observe where their graphs (loci) intersect; to solve them algebraically means to perform algebra operations on $y = x + 1$ and $y = 2x - 1$ until they turn into $x = 2$ and $y = 3$.

Geometry is like linear algebra except for two things: 1) The algebraic method is unavailable; we only look for the intersection of loci, and 2) Loci may be either lines or arcs. Linear algebra is usually the first proof-oriented math class that one takes in college; recalling that it was related to geometry even at the white-belt level will become the fondest memory of college students looking back on their high-school studies. I quote from *College Geometry* (p. 13) by Nathan Altshiller-Court, a standard lower-division university-level textbook:

> *In a great many cases, the solution of a geometric problem depends upon the finding of a point which satisfies certain conditions… If one of the conditions which the required point must satisfy be set aside, the problem may have many solutions. However, the point will not become arbitrary, but will move along a certain path, the geometric locus of the point. Now by taking into consideration the discarded condition and setting aside another, we make the required point describe another geometric locus. A point common to the two loci is the point sought.*

Teacher, you will do well to remember that there is attrition from both ends of the spectrum! Some white belts perform poorly because they need the material presented to them more slowly, which is why you must be capable of and have the patience to write a detailed two-column proof of the type displayed above. Capital-E Educators tend to assume that everybody who performs poorly is like this, which has resulted in some really tedious textbooks, though I believe that such students are in the minority. I tell you, there are also many poor performers who are not slow, as evidenced by their high marks in algebra, but who have concluded that geometry is wholly disconnected from algebra and can thus be jettisoned without loss. Capital-E Educators do not seem to realize how hectic teenagers' lives are with homework, sports, clubs, and mandatory "volunteer" work. They do not realize that teenagers prioritize.

That isosceles triangles have two angle bisectors equal is an easy corollary of ASA. Yellow belts proved the converse, but here we consider the classic proof by Jakob Steiner, which depends on the parallelogram theorem and is analogous to the proof of the mid-segment theorem, except that here J and K are the intersections of $\overline{EG}$ and $\overline{FG}$ with the bisectors of the opposite angles, where before we considered M_{GE} and M_{FG}. It is not satisfactory to use the parallel postulate to prove the converse of a theorem that does not require it but, since this was the only known proof for a long time, it is worth knowing, and it here serves our purpose in pedagogy.

Steiner–Lehmus Theorem

If a triangle has two angle bisectors equal, then it is isosceles.

Classic Proof

Given $\overline{EFG}$ with F^* and E^* the intersections of $\overline{EG}$ and $\overline{FG}$ with the bisectors of the opposite angles, given equal. Find the intersection, J, of loci $\overline{FJ} = \overline{FE^*}$ and $\overline{F^*J} = \overline{EF}$. Of the two, use the one on the G side of $\overleftrightarrow{EF}$. By SSS, $\overline{JFF^*} \cong \overline{FE^*E}$ and $\angle JFF^* = \angle FE^*E$. Also, $\angle JF^*F = \angle FEE^* = \angle F^*EE^*$, the former by congruence and the latter by angle bisection. Let $M := \overrightarrow{EE^*} \cap \overrightarrow{FF^*}$. $\angle EMF^* = \angle FME^*$ by the vertical angles theorem. By the angle sum theorem, the sum of the other two angles in $\overline{EMF^*}$ and $\overline{FME^*}$ are equal; that is, $\angle MF^*E + \angle F^*EM = \angle FE^*M + \angle MFE^*$. By the exterior angle theorem, this is $\angle EMF$ and it is obtuse because the other two angles of $\overline{EMF}$ are both half of the vertices of a triangle and so they must sum to an acute angle because, if they summed to an obtuse angle, double them would exceed a straight angle, in defiance of the angle sum theorem. $\angle JF^*F = \angle F^*EE^*$; so, substitute $\angle JF^*F$ for $\angle F^*EM$. $\angle JFF^* = \angle FE^*E$; so, substitute $\angle JFF^*$ for $\angle FE^*M$. $\angle MF^*E + \angle JF^*F = \angle JFF^* + \angle MFE^*$. These angles are obtuse so, by OSS, $\overline{EFJ} \cong \overline{JF^*E}$ and $\overline{FJ} = \overline{FE^*} = \overline{EF^*}$. By SSS, $\overline{EFE^*} \cong \overline{FEF^*}$, so $\angle EFE^* = \angle FEF^*$. By the isosceles triangle theorem converse, $\overline{EFG}$ is isosceles. ■

Those who excel in linear algebra will benefit from a teacher who can relate it to their study of geometry. Also of interest to algebraists are the concepts of analytic and synthetic knowledge, terminology first introduced by Immanuel Kant. Green belts learned that "Analytic is knowledge contained in the given information and analysis is just restating it in a different way, hopefully clearer. Auxiliary are lines or arcs not given whose intersection goes beyond analytic. For instance, in construction 1.1 we drew equal circles around J and K. Their intersection gives us knowledge of the apex L that is not contained in the information and is thus not analytic. But, for this knowledge to be synthetic, it must remain after the auxiliary lines and arcs are erased. There are a million lines and arcs that could potentially be added to a geometric diagram, but only the ones that leave relevant information after being erased are productive. If the information they provide only exists as long as they are in place, then that information is not solving the given problem, but solving a different one with additional given information."

Why is this of interest to those who are good at algebra? Let us perform algebra operations on $y = x + 1$ and $y = 2x - 1$ until they turn into $x = 2$ and $y = 3$. We set the right sides equal using the transitive property, add $1 - x$ to both sides, simplify to get $x = 2$, and then substitute this into either given equation to get $y = 3$. Adding $1 - x$ is the auxiliary, not the million other things that might have been added, because it leaves only the answer to the given question.

Thus, the teacher should be able to make it clear to good algebraists that geometry is doing essentially the same thing, only graphically. One obstacle yet remains. Some white-belt students have already attended and failed another geometry class and thus come with baggage. This is especially true if they did not fail but dropped out in protest of *Common Core*. Explain that this textbook does not use the SMSG axioms[120], whose Ruler Postulate states that any line can be placed one-to-one onto the real numbers so the distance between two points is the absolute value of the difference of their corresponding real numbers, and whose Angle Measurement Postulate states that every angle corresponds to a real number between 0 and 180. SMSG is apparently unaware that, by transitivity, this makes lengths and angles the same thing.

Glencoe Geometry (p. 256) declares two triangles congruent, one with all its sides and angles labeled: $a = 38.4$ mm, $b = 54$ mm, $c = 32.1$ mm and $\alpha = 45°$, $\beta = 99°$, $\gamma = 36°$. The other triangle has the side corresponding to a labeled $(x + 2y)$ mm and the angle corresponding to β labeled $(8y - 5)°$. *Glencoe* solves $x + 2y = 38.4$ and $8y - 5 = 99$ to get $x = 12.4$ and $y = 13$, mysteriously reported without unit labels. *What is the geometric interpretation of x and y???*

The chapter on congruent triangles in *Glencoe* is just one long lesson in the algebraic solution of simultaneous equations; the congruent triangles are used only to set up the two equations to be solved. They never solve three equations in three unknowns nor introduce Cramer's rule, making it all a very Algebra I lesson. The reason that nobody cares about the units of x and y is that nobody cares about geometry. "Angles, shmangles! Who cares what the units are as long as we do not have to learn anything new but can just review easy but boring algebra we already know?"

Educators with a capital E like the term "critical thinking" and use it often. At one time or another, just about every topic in every high-school subject has been described as instilling critical thinking in the students. It's a buzzword! Even geometry teachers are expected to teach students critical thinking. But what does this mean? Given the name, it must mean effective criticism.

Checking the units, as demonstrated in the preceding critique, is effective. But McGraw-Hill is not the only idiot in the room! *Geometry* by Houghton-Mifflin-Harcourt (p. 917) declares two triangles congruent; transferring all the lengths and angles to one triangle, we have $a = 11$ mm, $b = 8$ mm, $c = 14$ mm and $\alpha = 55°$, $\beta = 36°$, $\gamma = 89°$. By SSS, the triangle is fully defined. By the Law of Cosines, $\gamma = \text{acos}\left(\frac{a^2+b^2-c^2}{2ab}\right)$, so $\alpha \approx 51.6°$, $\beta \approx 34.8°$, $\gamma \approx 93.6°$. *Not even close!!!*

Common Core teachers just *hate* it when students use critical thinking on *them!* While they were not taking any college math classes, they were taking education classes about how to teach gifted students. They did not learn what to do about students who are smarter than they are. ☹

[120] A bastardization of Birkhoff: www.math.stonybrook.edu/~scott/mat515.fall14/smsg.pdf

Antiparallel Lines Should Be Called Supplementary Lines

I did not initially include a discussion of antiparallel lines in this book because I did not like the name. It seemed like it would confuse students, especially those for whom English is not their first language, because it translates to "not parallel." There are many lines that are not parallel to a given line and this naming convention does not single out the one under consideration.

But recently (11 Jan 23), a retired high-school teacher named Pat Ballew wrote a blog citing the *Wolfram Math* page about this, which gave the subject renewed interest and raised the specter that I might be thought a dummy who is unaware of this "advanced" geometry. So here goes!

Transversal Lemma ***(Euclid, Book I, Prop. 27)***

If alternate interior angles are equal, the two lines crossed by the transversal are parallel.

This appeared near the beginning of the orange-belt chapter, but it is actually neutral geometry (it was among the yellow-belt exit exam questions) because it depends only on the exterior angle inequality theorem. Euclid (*The Elements*, Book I, Prop. 16) and I both prove it using only neutral geometry; that is, without a parallel postulate. Let us now define supplementary lines as follows:

Lines, Supplementary	If alternate interior angles are supplementary, the two lines crossed by the transversal are supplementary relative to that transversal.

Wolfram, quoting Johnson ([1929] 2007, p. 172), rather clumsily puts it, "they make the same angle in the opposite senses." Maybe "opposite senses" was defined in 1929, but the only named angle pair in this book is alternate interior angles. This is red belt because the following theorems cite green- and red-belt theorems. I am not aware of any neutral geometry results about this.

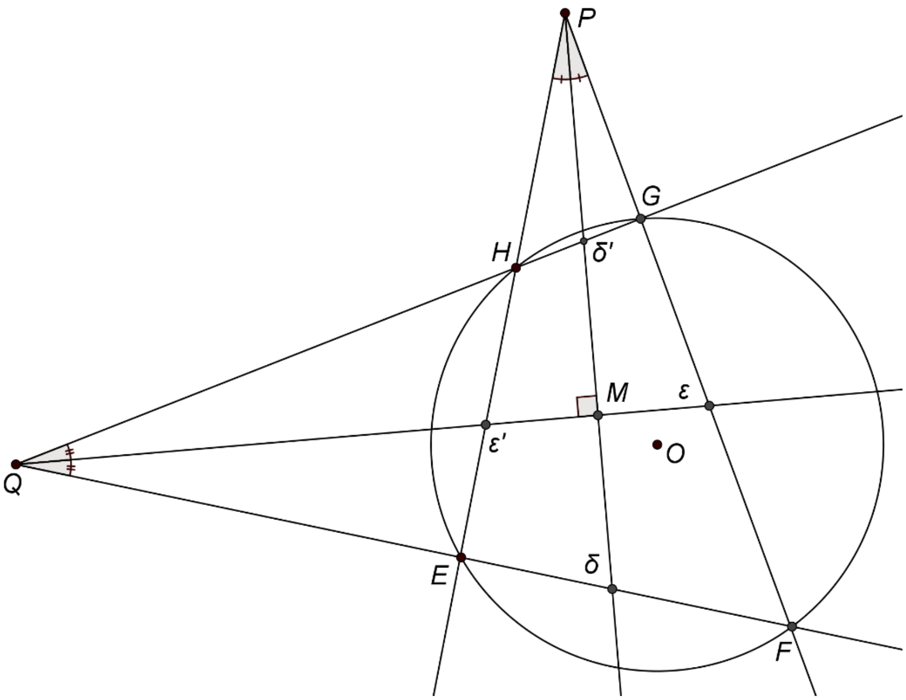
Supplementary Lines in Pairs Theorem

$\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are supplementary lines relative to the angle bisector of $\angle EPF$, as shown in the figure, if and only if $\overleftrightarrow{FG}$ and $\overleftrightarrow{HE}$ are supplementary lines relative to the angle bisector of $\angle FQG$.

Proof

Let the transversal be the bisector of $\angle EPF$. $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are supplementary lines relative to this transversal because $\delta + \delta' = \sigma$. Let $Q \coloneqq \overleftrightarrow{EF} \cap \overleftrightarrow{GH}$ and we will assume that it is on the E side of the angle bisector; if it is not, then relabel. Let M be the intersection of the $\angle P$ and $\angle Q$ bisectors. $\angle M$ is right by the center line theorem in the isosceles triangle with apex Q. By the center line theorem, M is the base midpoint of an isosceles triangle with apex P, so $\varepsilon + \varepsilon' = \sigma$. Thus, the result, and analogously for the converse. ■

Traditionally, the transversal being an angle bisector was baked into the definition, so the figure formed two nested similar triangles, $\overline{EFP}$ with angles α, β, γ, and $\overline{HGP}$ with angles β, α, γ. These are the only theorems we will prove here but, for us, the transversal is allowed to be other things.



Supplementary Lines are Cyclic Theorem

$\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are supplementary lines relative to the angle bisector of $\angle EPF$, as shown in the figure, if and only if $\overline{EFGH}$ is cyclic.

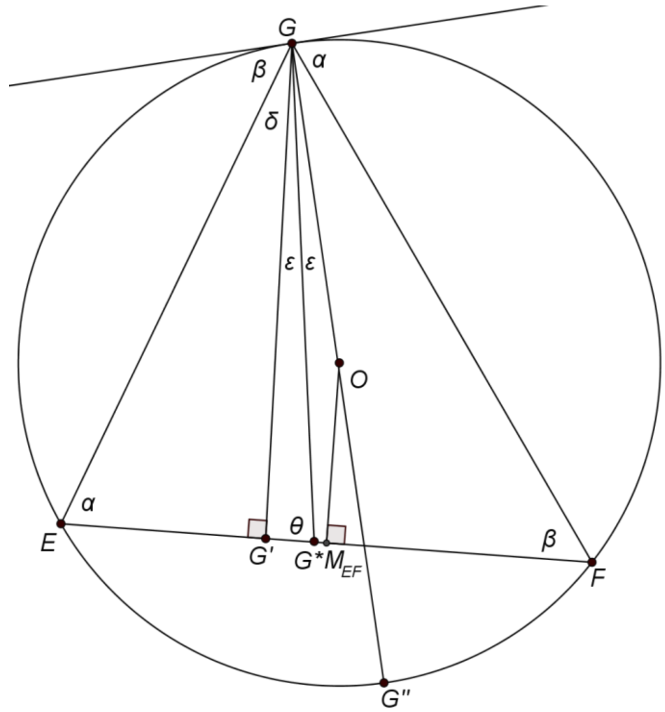
> ***Proof***
>
> Assume $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are supplementary. In the quadrilateral with opposite vertices E and M, by the quadrilateral angle sum theorem, $\angle E = \rho - \delta + \varepsilon$. In the quadrilateral with opposite vertices G and M, by the quadrilateral angle sum theorem, $\angle G = \rho + \delta - \varepsilon$. Thus, $\angle E + \angle G = \sigma$ and, by the cyclic quadrilateral theorem converse, $\overline{EFGH}$ is cyclic. •

Walk the proof backwards to get the converse.

Thus, if we center a circle on the base mediator so it goes through the base endpoints, the two cuts on the legs or their extensions define a line supplementary to the base. The circumcircle cuts both legs at the apex, in which case it is the tangent that is supplementary to the base (see below). In particular, the orthic triangle lemma informs us of one such circle:

Orthic Triangle Sides are Supplementary Theorem

Given $\overline{EFG}$, $\overleftrightarrow{EF}$ and $\overleftrightarrow{E'F'}$ are supplementary lines relative to the angle bisector of $\angle G$.



Base Supplementary to Apex Tangent Theorem
The tangent to the circumcircle touching at the apex is supplementary to the base relative to the apex angle bisector.

Proof
In the figure, the angles α equal each other and the angles β equal each other by the tangent and chord theorem. The angles ε equal each other by the skew angle theorem.

$\alpha - \beta = 2\varepsilon$	skew angle theorem
$\alpha = \rho - \delta$	complementary in $\overline{EG'G}$
$\rho - \delta = \beta + 2\varepsilon$	eliminate α from the above two equations
$\rho = \theta + \varepsilon$	complementary in $\overline{G'G^*G}$
$\theta = \beta + \delta + \epsilon$	eliminate ρ from the above two equations

Thus, the tangent to the circumcircle touching at the apex is supplementary to the base relative to the apex angle bisector. ■

By the transversal theorem, all the lines supplementary to the base of a triangle are parallel to each other because they all have the same alternate interior angle on the apex angle bisector; that is, $\sigma - \theta$ in the figure above. The line touching the circumcircle at the apex is supplementary to the base and, by the tangent theorem, it is perpendicular to the circumcircle radius to the apex, as are all the other lines supplementary to the base, by the transversal theorem corollary.

Foundations of Geometry Revisited

Patience Please!!!

I am still working on this three-page appendix. I left these pages blank now so I could update the page numbers in the index. The foundations appendix will be posted in the next month or two – it is now 1 Feb 23. I will announce it in the comments section at *Research Gate*.

www.researchgate.net/publication/291333791_Volume_One_Geometry_without_Multiplication

A Look Ahead: Blue Belt!!!

Young adult (YA) novelists typically end their stories with the hero and heroine narrowly escaping one death trap only to run headlong into another. This next adventure is described in a short sample chapter from the as-yet-to-be-written sequel. *Clever!* Borrowing a page from the YA playbook, I thought I would here prove a few theorems from the early pages of the blue-belt chapter, so students can see just how much fun *Volume Two* is going to be.

Medial Triangle Area Theorem
The medial triangle and the three triangles around it quarter the area of the parent triangle.

Medial Parallelogram Area Theorem I
The area of a medial parallelogram is half that of its parent quadrilateral.

Medial Parallelogram Area Theorem II

1. $|\overline{M_{EF}M_{EG}M_{GH}M_{FH}}| = \frac{1}{2}\left||\overline{EFG}| - |\overline{FGH}|\right| = \frac{1}{2}\left||\overline{EFH}| - |\overline{EGH}|\right|$ *(Part One)*
2. $|\overline{M_{FG}M_{FH}M_{HE}M_{EG}}| = \frac{1}{2}\left||\overline{EFG}| - |\overline{EFH}|\right| = \frac{1}{2}\left||\overline{FGH}| - |\overline{EGH}|\right|$ *(Part Two)*

The first case is E, H, M_{EG} on one side of $\overleftrightarrow{M_{EF}M_{GH}}$ and F, G, M_{FH} on the other side. The symbols $\frac{1}{4}$ and $\frac{3}{4}$ do not imply division; they denote the medial triangle area theorem quartering a triangle.

Proof of First Case

$|\overline{FGM_{GH}M_{FH}}| = \frac{3}{4}|\overline{FGH}|$ and $|\overline{EM_{EG}M_{GH}H}| = \frac{3}{4}|\overline{EGH}|$ and $|\overline{EM_{EF}M_{EG}}| = \frac{1}{4}|\overline{EFG}|$ and $|\overline{FM_{FH}M_{EF}}| = \frac{1}{4}|\overline{EFH}|$ by the medial triangle area theorem. Thus,

$$\begin{aligned}
|\overline{M_{EF}M_{EG}M_{GH}M_{FH}}| &= |\overline{EFGH}| - \tfrac{1}{4}\left(3|\overline{FGH}| + 3|\overline{EGH}| + |\overline{EFG}| + |\overline{EFH}|\right)\\
&= |\overline{EFGH}| - \tfrac{1}{4}\left(2|\overline{FGH}| + |\overline{FGH}| + 2|\overline{EGH}| + |\overline{EGH}| + |\overline{EFG}| + |\overline{EFH}|\right)\\
&= |\overline{EFGH}| - \tfrac{1}{2}\left(|\overline{FGH}| + |\overline{EGH}| + |\overline{EFGH}|\right)\\
&= \tfrac{1}{2}|\overline{EFGH}| - \tfrac{1}{2}\left(|\overline{FGH}| + |\overline{EGH}|\right)\\
&= \tfrac{1}{2}\left(|\overline{EFG}| + |\overline{EGH}|\right) - \tfrac{1}{2}\left(|\overline{FGH}| + |\overline{EGH}|\right)\\
&= \tfrac{1}{2}\left(|\overline{EFG}| - |\overline{FGH}|\right)
\end{aligned}$$

Now, let us replace the last two steps with this:

$$\begin{aligned}
&= \tfrac{1}{2}\left(|\overline{EFH}| + |\overline{FGH}|\right) - \tfrac{1}{2}\left(|\overline{FGH}| + |\overline{EGH}|\right)\\
&= \tfrac{1}{2}\left(|\overline{EFH}| - |\overline{EGH}|\right)
\end{aligned}$$

Part two is the same proof with different labels. •

The second case is E, H, M_{FH} on one side of $\overleftrightarrow{M_{EF}M_{GH}}$, and F, G, M_{EG} on the other side.

Proof of Second Case

$\left|\overline{FGM_{EG}M_{EF}}\right| = \frac{3}{4}\left|\overline{EFG}\right|$ and $\left|\overline{HM_{FH}M_{GH}}\right| = \frac{1}{4}\left|\overline{FGH}\right|$ and $\left|\overline{EM_{EF}M_{FH}H}\right| = \frac{3}{4}\left|\overline{EFH}\right|$ and $\left|\overline{GM_{GH}M_{EG}}\right| = \frac{1}{4}\left|\overline{GHE}\right|$ by the medial triangle area theorem. Thus,

$$\begin{aligned}\left|\overline{M_{EF}M_{EG}M_{GH}M_{FH}}\right| &= \left|\overline{EFGH}\right| - \tfrac{1}{4}\left(3\left|\overline{EFG}\right| + \left|\overline{FGH}\right| + 3\left|\overline{EFH}\right| + \left|\overline{GHE}\right|\right)\\ &= \tfrac{1}{2}\left(\left|\overline{FGH}\right| - \left|\overline{EFG}\right|\right) = \tfrac{1}{2}\left(\left|\overline{EGH}\right| - \left|\overline{EFH}\right|\right) \qquad \bullet \qquad \blacksquare\end{aligned}$$

The two cases show that we can always subtract the smaller area from the larger area. Once I set up the second proof, I skipped most of the steps because it is analogous with the first proof.

Medial Parallelogram Area Theorem III

Given $\overline{EFGH}$, assume $P := \overrightarrow{FG} \cap \overrightarrow{EH}$ exists; then $\left|\overline{EFGH}\right| = 4\left|\overline{M_{EG}M_{FH}P}\right|$.

Proof

By mid-segment theorem #1, $\overleftrightarrow{M_{FH}M_{GH}} \parallel \overleftrightarrow{FG}$. By mid-segment theorem #2, M_{FH}, M_{GH}, M_{PH} are collinear. By the triangle area and the medial triangle area theorem, $\left|\overline{M_{FH}M_{GH}P}\right| = \left|\overline{M_{FH}M_{GH}M_{FG}}\right| = \frac{1}{4}\left|\overline{FGH}\right|$.

By mid-segment theorem #1, $\overleftrightarrow{M_{EG}M_{GH}} \parallel \overleftrightarrow{EH}$. By mid-segment theorem #2, M_{EG}, M_{GH}, M_{PG} are collinear. By the triangle area and the medial triangle area theorems, $\left|\overline{M_{EG}M_{GH}P}\right| = \left|\overline{M_{EG}M_{GH}M_{HE}}\right| = \frac{1}{4}\left|\overline{EGH}\right|$.

By medial parallelogram area theorem II, $\left|\overline{M_{EF}M_{EG}M_{GH}M_{FH}}\right| = \frac{1}{2}\left(\left|\overline{EFG}\right| - \left|\overline{FGH}\right|\right)$. Half of this is $\left|\overline{M_{EG}M_{GH}M_{FH}}\right| = \frac{1}{4}\left(\left|\overline{EFG}\right| - \left|\overline{FGH}\right|\right)$.

$$\begin{aligned}\left|\overline{M_{EG}M_{FH}P}\right| &= \left|\overline{M_{FH}M_{GH}P}\right| + \left|\overline{M_{EG}M_{GH}P}\right| + \left|\overline{M_{EG}M_{GH}M_{FH}}\right|\\ &= \tfrac{1}{4}\left|\overline{FGH}\right| + \tfrac{1}{4}\left|\overline{EGH}\right| + \tfrac{1}{4}\left|\overline{EFG}\right| - \tfrac{1}{4}\left|\overline{FGH}\right|\\ &= \tfrac{1}{4}\left|\overline{EGH}\right| + \tfrac{1}{4}\left|\overline{EFG}\right| = \tfrac{1}{4}\left|\overline{EFGH}\right| \qquad \blacksquare\end{aligned}$$

Verify that this would have worked if we had set $\left|\overline{M_{EF}M_{EG}M_{GH}M_{FH}}\right| = \frac{1}{2}\left(\left|\overline{EFH}\right| - \left|\overline{EGH}\right|\right)$.

The only original result of *Geometry–Do* – Never seen before! – will now be proven. I just wish it were me who proved it – but it was my friend Milan Zlatanović of the University of Niš, Serbia.

Cramer–Castillon Problem
Given three non-collinear points inside a circle, construct a triangle with vertices on the circle and with a different side through each point.

Wholly unaware that this was an old-time problem, I just made it up while writing the green-belt chapter; I thought it might make a good exercise to keep the students busy over a long weekend!

I worked on it for months to no avail until I realized that my focus on it was distracting me from writing about the theorems that I did know how to prove – I might never finish the book!

At the time I was friends with a Russian geometry professor, so I thought I would show him the entrance to that rabbit hole – he went in headfirst! Every few months I would ask him for an update, and he would show me a dozen pages of *very* advanced algebra. His work seemed to have to do with elliptic curves though, frankly, it was over my head. Finally, I reminded him that I was writing a high-school textbook. If he succeeded – and that was still a big IF – it was beyond the scope of my book, and he should submit it to a math journal. He never spoke to me again.

In the meantime, I had become friends with a Serbian geometry professor, Dr. Zlatanović. I was really worried! Should I show him this problem and risk him never speaking to me again if he failed at it? Might I get a reputation as the wrecker of geometry professors' careers? With much trepidation, I decided to show Dr. Zlatanović the rabbit hole from which geometers never return.

Three days later, he had solved it. "That was a tough one," he said, "Send me another!"

Dr. Zlatanović's solution is blue belt because it makes use of the following quadrature theorems. Euclid proved these theorems in *Book III* by building rectangles on segments. He did not use proportions, which he introduced in *Book V*. But the proofs are easier if we have multiplication and the triangle similarity theorem; we will assume these so we can pretend to be blue belts. But we need not assume the intersecting chords/secants similarity theorems; they are green belt.

Intersecting Chords Theorem ***(Euclid, Book III, Prop. 35)***
If two chords of a circle intersect inside the circle, the product of the two segments of one is equal to the product of the two segments of the other.

> ***Proof***
> Given $\overline{EFGH}$ cyclic and T its bi-medial, by the intersecting chords similarity theorem, $\overline{EFT} \sim \overline{HGT}$. By the triangle similarity theorem, there exists k such that $|\overline{ET}| = k|\overline{HT}|$ and $|\overline{FT}| = k|\overline{GT}|$. Thus, $\frac{|\overline{ET}|}{|\overline{HT}|} = k = \frac{|\overline{FT}|}{|\overline{GT}|}$. By cross multiplication, $|\overline{ET}||\overline{GT}| = |\overline{FT}||\overline{HT}|$. ■

Intersecting Secants Theorem ***(Euclid, Book III, Prop. 36, 37)***

If two secants of a circle intersect outside the circle, the product of the two segments of one, from the intersection to where the circle cuts it, is equal to the product of the two segments of the other, from the intersection to where the circle cuts it.

Proof

Given $\overline{EFGH}$ cyclic and $P := \overrightarrow{FE} \cap \overrightarrow{GH}$, by the intersecting secants similarity theorem, $\overline{PFH} \sim \overline{PGE}$. By the triangle similarity theorem, there exists k such that $|\overline{PF}| = k|\overline{PG}|$ and $|\overline{PH}| = k|\overline{PE}|$. Thus, $\frac{|\overline{PF}|}{|\overline{PG}|} = k = \frac{|\overline{PH}|}{|\overline{PE}|}$. By cross multiplication, $|\overline{PE}||\overline{PF}| = |\overline{PH}||\overline{PG}|$. ■

Let P be inside a circle with center O and radius r. P cuts a chord into lengths x and y; it cuts a diameter z from O. By the intersecting chords theorem, $xy = (r+z)(r-z) = r^2 - z^2$. The **power of the point** is $r^2 - z^2$. Let P be outside the circle. A secant through it has lengths x and y to the first and second intersections with the circle, respectively. The secant through the center has lengths w and $w + 2r$ to the first and second intersections with the circle, respectively. By the intersecting secants theorem, $xy = w(w+2r) = w^2 + 2wr = w^2 + 2wr + r^2 - r^2 = (w+r)^2 - r^2 = z^2 - r^2$ with $z = |\overline{OP}|$. The power of the point is $|r^2 - z^2|$ for points both inside and outside the circle. The power of the point depends only on the point and the circle, not on the chord through P; it tells you the product xy even though you know neither x nor y.

If there is only one circle in a figure, the power of point P is denoted $|P|$. Vertical bars also denote absolute value, as $|x - y|$, but points are uppercase and lengths lowercase, so it should be clear. If there are multiple circles, one must specify to which circle the power of a point is defined. In the solution to the Cramer-Castillon problem below, ω is the circle that defines powers of points.

The power of the point is a difference of squares, so we need to construct a square of this size. In the lemma, x represents the larger of r and z, and y represents the smaller of r and z.

Lemma 5.9 ***(Euclid, Book II, Prop. 5)***

Given lengths $y < x$, the rectangle of sides $x + y$ and $x - y$ is equal in area to the square of side x minus the square of side y.

Proof

Construct the small square $\overline{EFGH}$ in the corner of the big square, $\overline{EJKL}$, with F inside $\overline{EJ}$ and H inside $\overline{EL}$. Extend $\overrightarrow{HG}$ to intersect $\overline{JK}$ at M. $\overline{EJKL} - \overline{EFGH} = \overline{HMKL} \cup \overline{JMGF}$. Construct $\overline{MNOK}$ congruent to $\overline{JMGF}$ outside $\overline{HMKL}$. $\overline{HNOL} = \overline{HMKL} \cup \overline{MNOK}$ is a rectangle. By transitivity, $\overline{HNOL} = \overline{EJKL} - \overline{EFGH}$, so $|\overline{HNOL}| = |\overline{EJKL}| - |\overline{EFGH}|$. ■

Construction 5.2 ***(Euclid, Book II, Prop. 14)***

Given two squares, construct a square equal in area to their difference.

Solution

Using the figure of lemma 5.9, extend $\overrightarrow{HN}$ by the length $\overline{NO}$ and label this point P so $\overline{PN} = \overline{NO}$. Build a semicircle on $\overline{HP}$ on the same side as J. Extend $\overrightarrow{ON}$ until it meets this semicircle at Q. A square built on $\overline{NQ}$ is equal in area to $\overline{HNOL} = \overline{EJKL} - \overline{EFGH}$. ■

Proof

Connect $\overline{PQ}$ and $\overline{QH}$. By Thales' diameter theorem, $\overline{PQH}$ is a right triangle. By the right triangle theorem, $\overline{PNQ} \sim \overline{QNH}$. By the triangle similarity theorem, $\frac{|\overline{PN}|}{|\overline{NQ}|} = \frac{|\overline{NQ}|}{|\overline{HN}|}$. By cross multiplication, $|\overline{NQ}||\overline{NQ}| = |\overline{PN}||\overline{HN}|$. $|\overline{NQ}||\overline{NQ}| = |\overline{NO}||\overline{HN}| = |\overline{EJKL}| - |\overline{EFGH}|$. These equalities are by substitution and lemma 5.9, respectively. ■

As an aside, C. 5.2 can be used to find the square root of a number, n, represented by the length of a segment. Extend it by one unit and build a semicircle on this $(n + 1)$–length segment, then raise a perpendicular from the endpoint of the n–length segment and measure its length to where it cuts the semicircle. Suppose I want $\sqrt{5}$. I draw a segment six units long, put my compass pin at the three-unit mark to draw the semicircle, then raise a perpendicular from the five-unit mark and measure its distance to the semicircle. Accuracy depends on the size of your compass.

Construction 5.3

Given an angle and a point inside a circle, draw a chord through the point that subtends the angle.

Solution

Given P in a circle with center O and radius r, replicate the angle in an equal circle and draw the chord that subtends it. Build a semicircle on this chord. The power of the point is $|P| = r^2 - z^2$ with $z = |\overline{OP}|$. By C. 5.2, find a length p such that $p^2 = r^2 - z^2$. Draw a line p distant from the chord on the same side as the semicircle. From where it cuts the semicircle, drop a perpendicular to the chord. The foot of this perpendicular cuts the chord as P cuts the desired chord. Circles around P of these radii (the cut segments of the chord in the equal circle) cut the given circle at the chord endpoints. ■

Discussion

For a given point there is no maximum angle but, by the shortest chord theorem, the chord through P perpendicular to the diameter through P subtends the smallest possible angle. If this is bigger than the given angle, then the construction is impossible. ■

And now, what you have all been waiting for, the solution to the Cramer-Castillon problem!

Solution

Let P_1, P_2, P_3 be points inside a circle ω with center O and radius r; the desired triangle is $\overline{EFG}$ with $P_1 \in \overline{EF}$ and $P_2 \in \overline{FG}$ and $P_3 \in \overline{GE}$. We first assume that $\overline{EFG}$ is known (draw $\overline{EFG}$ and then P_1, P_2, P_3) and draw the figure described in the next paragraph. Then we will determine which points can be derived just from P_1, P_2, P_3 and not from E, F, G. Then we will show that E, F, G can be derived from these points. This completes the proof.

Find J on the circle such that $\overleftrightarrow{FJ} \parallel \overleftrightarrow{P_2P_3}$ and let K be the intersection of $\overleftrightarrow{EJ}$ and $\overleftrightarrow{P_2P_3}$. Find L on the circle such that $\overleftrightarrow{FL} \parallel \overleftrightarrow{P_1K}$ and let M be the intersection of $\overleftrightarrow{JL}$ and $\overleftrightarrow{P_1K}$. Case #1 is that J is on the arc of $\overline{EG}$. Case #2, J on the arc of $\overline{EF}$, requires replacing two sentences, which are shown in red after the sentences they replace. $J \equiv E$ is left as an exercise.

$\angle EKP_2 = \angle EJF = \angle EGP_2$ by T & V and the inscribed angle theorem, respectively. $\angle EKP_2 + \angle EJF = \sigma = \angle EJF + \angle EGP_2$ by T & V and the cyclic quadrilateral theorems, respectively. $\overline{P_2GKE}$ is cyclic by the inscribed angle theorem converse. By the intersecting chords theorem, $|\overline{P_3P_2}||\overline{P_3K}| = |\overline{P_3E}||\overline{P_3G}|$. But $\overline{EG}$ is also a chord in ω so $|\overline{P_3E}||\overline{P_3G}|$ is the power of point P_3 relative to ω, which we call $|P_3|$. Thus, $|\overline{P_3K}| = \frac{|P_3|}{|\overline{P_3P_2}|}$ and K can be constructed. $\angle EJM + \angle EFL = \sigma$ by the cyclic quadrilateral theorem, but $\angle EP_1M = \angle EFL$ by T & V, so $\angle EJM + \angle EP_1M = \sigma$ and $\overline{EP_1MJ}$ is cyclic by the cyclic quadrilateral theorem converse. $\angle EJM = \angle EFL$ by the inscribed angle theorem, but $\angle EP_1M = \angle EFL$ by T & V, so $\angle EJM = \angle EP_1M$ and $\overline{EJP_1M}$ is cyclic by the inscribed angle theorem converse. By the intersecting secants theorem, $|\overline{KM}||\overline{KP_1}| = |\overline{KJ}||\overline{KE}|$. But $\overline{JE}$ is also a chord in ω so $|\overline{KJ}||\overline{KE}|$ is the power of point K relative to ω, which we call $|K|$. Thus, $|\overline{KM}| = \frac{|K|}{|\overline{KP_1}|}$ and M can be constructed. K and M can be constructed, so $\angle P_3KM$ can be constructed. But $\angle JFL = \angle P_3KM$ by the parallelogram angles theorem, so we know $\angle JFL$. By C. 5.2, draw through M a chord $\overline{JL}$ in ω that subtends this angle. Thus, J, K, L, M can all be constructed independent of E, F, G. $\overleftrightarrow{KJ}$ intersects ω at E. A line through L parallel to $\overleftrightarrow{KM}$ intersects ω at F. $\overleftrightarrow{EP_3}$ intersects ω at G, as does $\overleftrightarrow{FP_2}$. ■

This problem has never before appeared in a textbook because Castillon's solution was too long, complicated and advanced for pedagogic purposes. So, let us call this the Zlatanović problem!

Now, hopefully, the students will all spend the next year anxiously awaiting some fulfillment from Amazon when their back-ordered copy of *Geometry with Multiplication* is finally delivered.

Needful Things

Students going to college without having studied blue belt should review *Elementary Quadrature Theory*[121] and this section, where we discuss the most needful similarity theorems.

Extant terminology regarding similarity is ill-conceived in two ways that often confuse students:

1. The point where a vertex angle bisector cuts the opposite side of a triangle has no name.
2. Angle bisector theorem has different meanings in beginner and intermediate textbooks.

A principal innovation of mine – perhaps the only thing I will be remembered for after my death – is denoting the midpoint of $\overline{EF}$ as M_{EF}. This makes geometry so much easier than it is in other textbooks, where midpoints are assigned randomly selected letters! Look at the proofs of the medial parallelogram area theorems II and III in the preceding appendix. Imagine what these proofs would be like if all the segment midpoints were assigned random letters. *Confusing!!!*

In this same spirit of clarity, I always represent the foot of an altitude with a single apostrophe; that is, E', F', G' are the feet of altitudes dropped from E, F, G, respectively. I never use single apostrophes for any other points, which is why I sometimes have E'', F'', G'' in a proof (*e.g.,* the Napoleon theorem) even if there is no mention of E', F', G'.

Sadly, there is no word comparable to "foot" for the point where an angle bisector cuts the opposite side of a triangle. In other textbooks, there is no consistent notation either; they denote these points with random letters. In the notation section at the beginning of this book, I fix the labeling problem by calling these points E^*, F^* and G^*. But I think we need a name that can be spoken aloud in lectures, so I am going to call them the infeet of a triangle; individually, an **infoot**.

The following theorem is seen as needful to first-year geometers.

Infoot Ratio Theorem ***(Euclid, Book VI, Prop. 3)***

The infoot cuts the base in the ratio of the legs, and the converse. For G^* of $\overline{EFG}$, $\frac{|\overline{EG^*}|}{|\overline{FG^*}|} = \frac{|\overline{EG}|}{|\overline{FG}|}$.

Proof is easy; draw a parallel to $\overleftrightarrow{FG}$ through E that cuts the bisector of $\angle G$. Intermediate textbooks call this the angle bisector theorem while beginner textbooks follow my practice of the angle bisector theorem being that angle bisectors are equidistant from the sides. This would be confusing for students of *Volume Two*; thus, blue belts need this new name for this old theorem.

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[121] Solve P. 5.1 without Miquel. Also, prove that, if P is the intersection of two disjoint circles' external tangents, then, for any secant from P that cuts the circles at consecutive points E, F, G, H, $|\overline{PE}| \times |\overline{PH}| = |\overline{PF}| \times |\overline{PG}|$.

Let us call it the **exfoot** where the bisector of an exterior angle of a triangle cuts the extension of the opposite side; label the exfeet $E^\times, F^\times$ and $G^\times$. An analogous theorem holds.

Exfoot Ratio Theorem
The exfoot cuts the extension of the base in the ratio of the legs, and the converse. For exfoot $G^\times$ of $\overline{EFG}$, $\frac{|\overline{EG^\times}|}{|\overline{FG^\times}|} = \frac{|\overline{EG}|}{|\overline{FG}|}$.

Euclid only proved the former. According to Euclid's translator, Heath, the latter was proven by Simson. Simson is the man who was falsely credited with what we now call the Wallace line. But Simson is renowned, and it would be sad if the only mention of him in this book is to discredit him, so we will here give him credit for something he did do. The infoot and the exfoot define **harmonic division**. It is college geometry; in high school, only the infoot theorem is needed. But, just in case you get a professor who thinks harmonic division is still taught in high school, as it was in Wentworth's day, you should know that, by transitivity, $\frac{|\overline{EG^*}|}{|\overline{FG^*}|} = \frac{|\overline{EG^\times}|}{|\overline{FG^\times}|}$. In words, you say that G^* and $G^\times$ divide $\overline{EF}$ internally and externally, respectively. Because the ratios are equal, they together divide $\overline{EF}$ harmonically. The figure $\overline{EFG}$ with G^* and $G^\times$ located is called a **pencil**.

Thales' diameter theorem is proven at the very beginning of green belt and is thus second-year *Geometry–Do*, though students who fail to get a green belt are advised to go a few pages into green belt on their own because this and the inscribed angle theorem on the next page are generally expected of all high-school graduates. In blue belt is another theorem due to Thales. Clarity demands that the adjectives "diameter" or "proportionality" be used to avoid confusion, though some textbooks say Thales' theorem for the former and intercept theorem for the latter.

Thales' Proportionality Theorem
The sides of an angle cut by some parallel lines are divided into proportional segments.

This is the general case of the side-splitter theorem, but with more parallel lines. The converse is also true; proportional segments imply that the lines are parallel. The following theorem is not really a corollary because it does not cite Thales' proportionality theorem, but it is closely related.

Thales' Proportionality Theorem Corollary
Parallel lines cut by some angles with the same vertex are divided into proportional segments.

Euclid put the Pythagorean theorem in Book I and triangle similarity in Book VI, which is also the organization of *Geometry–Do*. Geometry books that are really algebra books begin with similarity and prove the Pythagorean theorem with the triangle similarity theorem and cross multiplication.

Orange belts prove the right triangle theorem, that the altitude to the hypotenuse of a right triangle forms two triangles similar to it and to each other. Let the legs be u, v; the hypotenuse be w; the altitude to the hypotenuse be h; and the projections of u and v on w be u' and v'.[122]

Pythagorean Theorem (Algebra Version)
$u^2 + v^2 = w^2$

Proof
$\frac{u'}{u} = \frac{u}{w}$ and $\frac{v'}{v} = \frac{v}{w}$ by the right triangle and triangle similarity theorems. $u^2 = wu'$ and $v^2 = wv'$ by cross multiplication. Add these to get $u^2 + v^2 = wu' + wv' = w^2$. ■

Proportions are something you should have learned in Algebra I; college professors will expect you to be rock solid on this. Just in case you are a little wobbly, we will review now. Also, there is geometric terminology that students should know, but it is not always taught in algebra class.

Before typesetters had fractions, proportions were written $a : b = c : d$. One says that a is to b as c is to d. The extremes are a and d and the means are b and c. $a : b = c : d$ implies $\frac{a}{b} = \frac{c}{d}$ but the converse is not true because lengths must be positive. When geometers use fractions, they tacitly assume $a, b, c, d \in \mathbb{R}^+$. If you multiply both sides of $\frac{a}{b} = \frac{c}{d}$ by bd, you get $ad = bc$. This is called cross multiplication. If the means are equal, that value is called the mean proportional, which is the same thing as the geometric mean; $\frac{a}{b} = \frac{b}{c}$ implies $b = \sqrt{ac}$. If one of the extremes is the unknown quantity, it is called the third proportional; $a = \frac{b^2}{c}$ or $c = \frac{b^2}{a}$. In this case, the ratio of the extremes is $\frac{a}{c} = \frac{a^2}{b^2} = \frac{b^2}{c^2}$. If the means are not equal and one of the extremes is the unknown quantity, it is called the fourth proportional; $\frac{a}{b} = \frac{c}{d}$ implies $a = \frac{bc}{d}$ or $d = \frac{bc}{a}$.

The geometric method for finding the fourth proportional cites Thales' proportionality theorem. Lay off a and then b on one ray, c on the other and then draw a parallel through the end of b. Third proportional means c in $a : b = c : d$. To find the mean proportional, b, use Thales' diameter theorem; $a + c$ is the diameter and you raise a perpendicular from their joint.

Right Triangle Theorem Corollaries ***(Euclid, Book VI, Prop. 13)***

1. *The right vertex altitude is the **geometric mean** of the projections;* $h = \sqrt{u'v'}$.
2. *Each leg is the geometric mean of the leg's projection and the hypotenuse;* $u = \sqrt{u'w}$.
3. *The product of the altitude and the hypotenuse is the product of the legs;* $hw = uv$.

[122] Never use a, b, c for the sides of a right triangle; it is confusing if you then use it to set up $ax^2 + bx + c = 0$.

Given $\frac{a}{b} = \frac{c}{d}$ the following are named operations that you should be able to do in a single step:

Inversion	$\frac{b}{a} = \frac{d}{c}$	Flip both fractions upside down.
Alternation	$\frac{a}{c} = \frac{b}{d}$ or $\frac{d}{b} = \frac{c}{a}$	Switch the means or switch the extremes.
Composition	$\frac{a \pm b}{b} = \frac{c \pm d}{d}$	Add or subtract the denominators to the numerators.
Series Sum	$\frac{a}{b} = \frac{c}{d} = \frac{a \pm c}{b \pm d}$	For a series of ratios, be sure the + and – line up.

A triangle's area is $A = \frac{eh_E}{2} = \frac{fh_F}{2} = \frac{gh_G}{2} = \frac{bh}{2}$, the latter if one side is the base. $A = rs$ is easy. If two triangles have the same height, alternation yields the same-height proportion, $\frac{A_1}{A_2} = \frac{b_1}{b_2}$. Given $\overline{EFG}$, if rays from E, F, G are concurrent at an interior point, P, we will call where they cut the opposite sides E^P, F^P, G^P, respectively. Example: Prove that $\frac{|\overline{EPG}|}{|\overline{FPG}|} = \frac{\overline{EG^P}}{\overline{FG^P}}$. By the same-height proportion, $\frac{|\overline{EG^PG}|}{|\overline{FG^PG}|} = \frac{|\overline{EG^PP}|}{|\overline{FG^PP}|} = \frac{\overline{EG^P}}{\overline{FG^P}}$; by series sum, $\frac{|\overline{EG^PG}|-|\overline{EG^PP}|}{|\overline{FG^PG}|-|\overline{FG^PP}|} = \frac{|\overline{EPG}|}{|\overline{FPG}|} = \frac{\overline{EG^P}}{\overline{FG^P}}$. The same-height proportion solves problems where a triangle is inscribed in a figure and its vertices cut the sides at given ratios. If $|\overline{EFG}| = 84$ m^2 and $\overline{EX} = 2\overline{FX}$ and $\overline{GY} = 3\overline{FY}$, find $|\overline{XFY}|$. It is $\frac{84}{3 \times 4} = 7$ m^2.

Incenter Ratio Theorem

The incenter cuts the bisector of an angle as the sum of its adjacent sides is to its opposite side.

And that is really all the work with similarity that is expected of American high-school graduates. Relentless use of cross multiplication, inversion, alternation, composition, and series sum of proportions will smoke the SAT. But, a word of warning: I checked the high-school math curriculum in Serbia and found that Ceva's, Menelaus', Stewart's and Ptolemy's theorems – all blue belt – are mandatory, and the former is used to prove the existence of the Gergonne point. I am told that, in practice, the latter two are taught only in the best high schools, or to honor students in the general Serbian high schools. But *every* Serbian – even those not college bound – learns of Ceva and Menelaus and applies their theorems to practical problems.

Every Serbian boy aspires to be another Nikola Tesla. If you do not want a Serb to kick your butt four years from now when you go out into the big world with an electrical engineering degree from an American university, then you had better buy *Geometry with Multiplication* and read the blue-belt chapter on your own, even if it is not required to graduate from an American high school. *Common Core* institutionalized mediocrity, but that is no excuse for *you* to be mediocre!

Preparation for Altshiller-Court's *College Geometry*

College Geometry, by Nathan Altshiller-Court, was the standard American undergraduate geometry textbook until colleges abandoned geometry due to the incoming freshmen not being even close to capable of reading the first chapter. But, in 2007, Dover snatched it from the jaws of obscurity, so it is now available again as a college geometry textbook, at least until I publish *Volume Two* of *Geometry–Do*, which will compete with it. But the first step towards competing successfully for the college textbook market is for there to be one. This requires that high school graduates be capable of reading the first chapter of *College Geometry*, or any college textbook. Altshiller-Court helpfully lists (pp. 1-2) nine constructions that he expected of any high-school graduate in 1952. Let us have a look! With the triangle similarity theorem, we have got this.

1. Divide a given segment into a given number of equal parts.

   Constructions 1.2 and 3.11 bisect and trisect a segment, respectively. Three applications of C. 1.2 quadrisect a segment. For integers $5 \leq n$, use a modification of C. 3.9. Given $\overline{OE_n}$, draw a ray $\overrightarrow{OG_1}$ and then lay off $\overline{OG_1} = \overline{G_1G_2} = \cdots = \overline{G_{n-1}G_n}$ on it.

2. Divide a given segment into a given ratio (i) internally; (ii) externally.

   Solution
   Given $\overline{EF}$ and lengths p and q in the desired ratio, draw parallel lines through E and F. Lay off p and q so $\left|\overline{EP}\right| = p$ and $\left|\overline{FQ}\right| = \left|\overline{FQ''}\right| = q$ with P and Q on opposite sides of $\overleftrightarrow{EF}$. $\overleftrightarrow{PQ}$ cuts $\overline{EF}$ internally and $\overleftrightarrow{PQ''}$ cuts $\overrightarrow{EF}$ externally. ■

   Proof
   Let G and H be the internal and external cuts, respectively. By the crossed triangle and triangle similarity theorems, $\overline{EPG} \sim \overline{FQG}$, which holds the ratio $\frac{p}{q} = \frac{|\overline{EG}|}{|\overline{FG}|}$. By the nested triangle and triangle similarity theorems, $\overline{EPH} \sim \overline{FQ''H}$, which holds the ratio $\frac{p}{q} = \frac{|\overline{EH}|}{|\overline{FH}|}$. ■

3. Construct the fourth proportional to three given segments.

   Solution
   Given $\frac{a}{b} = \frac{c}{d}$ with d unknown, draw an angle and lay off a and then b on one ray and c on the other. Connect the ends of a and c and then draw a parallel to this through the end of b. It cuts the other ray at the end of d. ■

Proof

By Thales' proportionality theorem. ■

If $b = c$, then the unknown d is called the third proportional. If c is wanted, invert to get $\frac{b}{a} = \frac{d}{c}$.

4. Construct the mean proportional to two given segments.

Solution

Given $\frac{a}{b} = \frac{b}{c}$ with b unknown, draw a segment $a + c$, bisect it and draw a semicircle around the midpoint. Raise a perpendicular from the a to c joint; its height is b. ■

Proof

By Thales' diameter theorem and right triangle theorem corollary #1. ■

5. Construct a square equal in area to a given (i) rectangle; (ii) triangle.

a. This is equivalent to problem #4 because $b^2 = ac$ implies that $\frac{a}{b} = \frac{b}{c}$.

ii. Choose a triangle side to be the base, a, and let c be half the apex altitude. Then (i).

6. Construct a square equal in area to the sum of two, three, or more given squares.

Pythagorean theorem for the first two, then repeat for additional squares.

7. Construct two segments given their sum and their difference.

Construction 2.3.

8. Construct the tangents from a given point to a given circle.

Construction 4.4, or construction 2.2 if you are not a green belt yet.

9. Construct the internal and external tangents of two given circles.

Constructions 3.12 and 3.13.

College Geometry begins (pp. 3-9) with five problems. #5 and #6 test your ability to spot the two similar triangles and to divide a segment into a given ratio. #7 is harder, but analogous. #8 is P. 3.51, about parallelograms. #9 is P. 3.52; easy once you think of the equal perpendiculars theorem. Any orange belt can do these problems if he also knows the triangle similarity theorem.

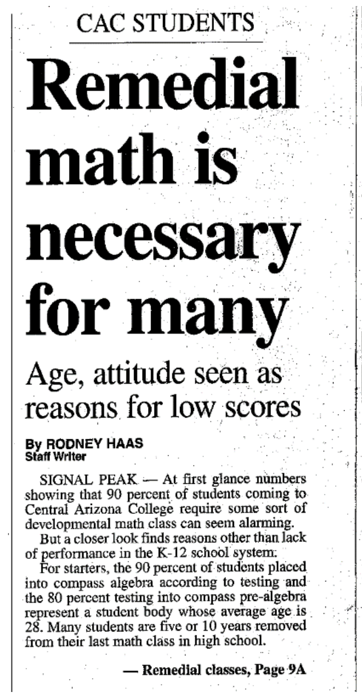
Geometry Jokes and Puzzles

Q: What is sad about Euclidean geometry? (I mean, besides the F you got?)
A: Parallel lines. They have so much in common, yet they will never meet.

Q: Why are acute rectangles regarded with suspicion?
A: They are just not right.

Q: Why is geometry so much more difficult than sociology?
A: Because triangles have three sides; people have only one.

Q: What is funny about this newspaper article?[123]
A: The remedial class is not because it has been "five to ten years" since the 28-year-old took a math class at the age of 16. It's because he's a moron!



CAC STUDENTS

Remedial math is necessary for many

Age, attitude seen as reasons for low scores

By RODNEY HAAS
Staff Writer

SIGNAL PEAK — At first glance numbers showing that 90 percent of students coming to Central Arizona College require some sort of developmental math class can seem alarming.

But a closer look finds reasons other than lack of performance in the K-12 school system.

For starters, the 90 percent of students placed into compass algebra according to testing and the 80 percent testing into compass pre-algebra represent a student body whose average age is 28. Many students are five or 10 years removed from their last math class in high school.

— Remedial classes, Page 9A

Q: What does a *Common Core* geometry student get when he adds a length and an angle?
A: A lengle!

Q: David Conley boasts, "International comparisons also helped ensure the [*Common Core*] standards were set at a high level." In which country did he do his comparison?
A: Lengle Land!

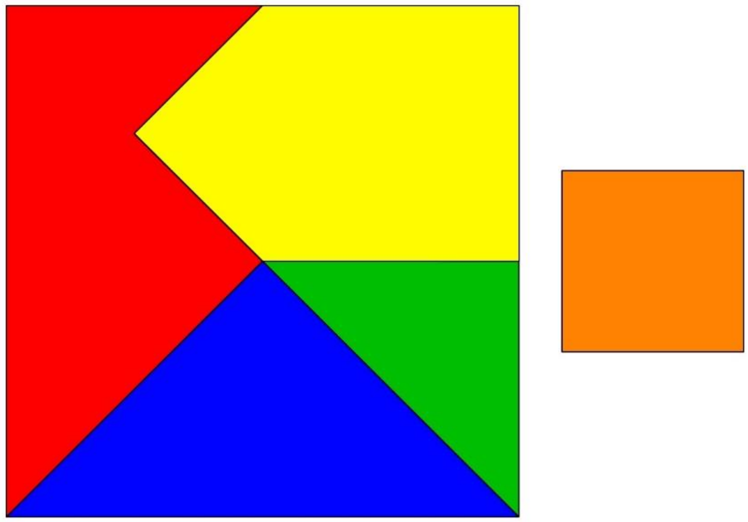
Q: How will a *Common Core* geometry student win the *International Mathematical Olympiad*?
A: Pineapples don't have sleeves.

A *Common Core* geometry teacher must substitute for the science teacher. He tells the students, "Remember, water boils at 90 degrees." But a student corrects him, "Teacher, you are mistaken! Water boils at 100 degrees." The geometry teacher consults his *Common Core* textbook and replies, "You are correct. It is the right angle that boils at 90 degrees!"

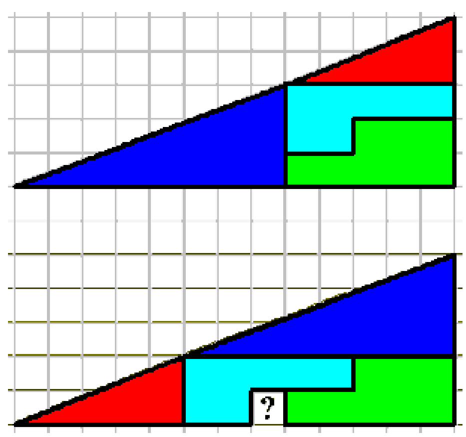
A Chukcha[124] is hunting seals when an American submarine surfaces. The captain asks, "Which way to Alaska?" The Chukcha points with his finger and the captain shouts down into his submarine, "Bearing, south 22° east!" They submerge. An hour later, a Russian submarine surfaces. The captain asks, "Which way did the Americans go?" The Chukcha replies, "Bearing, south 22° east." The Russian captain pleads, "Don't be a wise guy. Just point with your finger!"

[123] *Casa Grande Dispatch*, 24 August 2016, front page; published in Casa Grande, Arizona, U.S.A.
[124] The Chukchi are the same race as the Inuit, but they live on the Russian side of the Bering Strait.

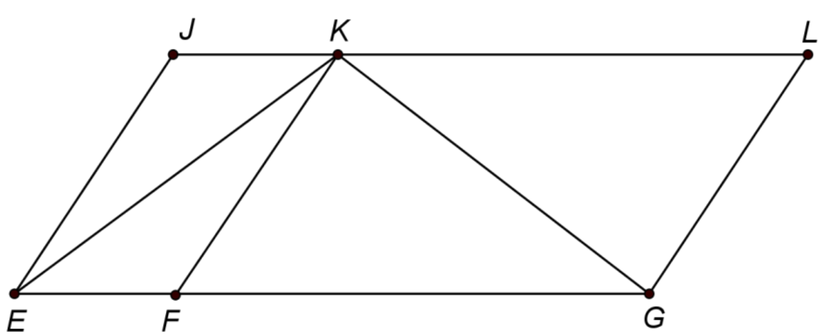
Photocopy the page, cut out the five pieces and assemble them into one big square.



Cut out the colored pieces in the upper triangle and reassemble them into the lower triangle. There's a hole! Should we notify the Physics Department that we've discovered an exception to their so-called "law" of the conservation of mass? Or is there a geometric explanation for this?



In the following figure, is $\overline{EK}$ less than, equal to or greater than $\overline{GK}$?



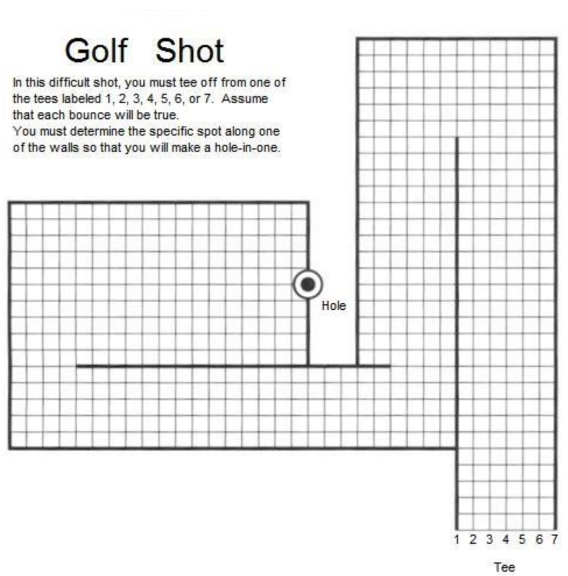
From Secret Place to Crossbones Rock,
Pace out what steps you may.
Turn right at rock and pace the same,
And you'll have found point A.

Return to Secret Place and count,
Your steps to Hangman's Tree.
Turn left at tree and now count down,
To take you to point B.

Halfway between points A and B,
You'll find my treasure case,
But what a shame that you can't know,
About my Secret Place.

When the young chieftain first married, he gave his bride a teepee made of buffalo hide. When he took a second wife, he gave her a bearskin teepee. Then, when he became Big Chief, he married the most beautiful maiden of all and gave her a two-story teepee of hippopotamus hide. But then he became ill and, realizing that he was dying, he told all the young braves that, if anyone could explain why his beautiful third wife got a two-story teepee of hippopotamus hide, he would make him the Big Chief. One brave stepped forward with the explanation. What was it?[125]

Who says geometry is impractical? You can make big money as a miniature golf hustler! 😉



[125] The sum of the squaws of the two hides is equal to the squaw of the hippopotamus.

Index of Postulates, Theorems and Constructions

Euclid's Postulates Plus One More

Segment	Two points fully define the segment between them.
Line	By extending it, a segment fully defines a line.
Triangle	Three noncollinear points fully define a triangle.
Circle	The center and the radius fully define a circle.
Right Angle	All right angles are equal; equivalently, all straight angles are equal.
Parallel	A line and a point not on it fully define the parallel through that point.

Equivalence Relations and Total Orderings

A relation is an operator, R, that returns either a "true" or a "false" when applied to an ordered pair of elements from a given set. Relations must be applied to objects from the same set. There are four ways that relations may be characterized:

Transitive	$a\ R\ b$ and $b\ R\ c$ implies $a\ R\ c$
Reflexive	$a\ R\ a$
Symmetric	$a\ R\ b$ implies $b\ R\ a$
Anti-Symmetric	$a\ R\ b$ and $b\ R\ a$ implies $a = b$

A relation that is reflexive, symmetric, and transitive is called an equivalence relation. The equivalence relations considered in geometry are equality, $=$; congruence, $\cong$; similarity, $\sim$; and, in Euclidean geometry but not in non-Euclidean geometry, parallelism, $\parallel$.

A relation that is reflexive, anti-symmetric and transitive is called an ordering. Geometers only consider one: less than or equal to, $\leq$. An ordering is total if $a \leq b$ or $b \leq a$, always. A set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a magnitude.

Additive Groups

We define an additive group as a set that is closed under an operation that we will denote $+$ and which has these properties:

Associative property	$(a + b) + c = a + (b + c)$
Commutative property	$a + b = b + a$
Existence and uniqueness of an identity	$a + 0 = a = 0 + a$
Existence of unique inverses (identity is its own)	$a + (-a) = 0 = (-a) + a$

The symbol **MD** denotes all the theorems and constructions in the 691-page *Geometry* by Moise and Downs ([1964] 1991). The symbol **WP** denotes all the theorems and constructions in the 80-page *Chapter IX, Geometry, Practical Shop Mathematics* by Wolfe and Phelps ([1935] 1958). Both are a survey of basic geometry before the 1960s counterculture when geometry was dumbed down to keep kids in school; later, *Common Core* made it worse. Wolfe and Phelps write (p. v):

> *The authors have kept in mind its use not only in factory schools, trade schools, vocational high schools,* etc.*, but also in all high schools to replace the usual geometry course for those students not intending to go to college… [It is] of much greater value to the high school student who is not going to college than is the usual geometry course consisting of about 150 theorems.*

Replication Axiom **4**
Given $\overline{EF}$ and $\overrightarrow{JK}$, there exists a unique point L on $\overrightarrow{JK}$ such that $\overline{EF} = \overline{JL}$.
Given $\angle EFG$ and $\overrightarrow{KJ}$, there exist rays $\overrightarrow{KL}$ and $\overrightarrow{KL''}$ such that $\angle EFG = \angle JKL = \angle JKL''$.

Interior Segment Axiom **4**
If M is between E and F, then $\overline{EM} < \overline{EF}$ and $\overline{MF} < \overline{EF}$ and $\overline{EM} + \overline{MF} = \overline{EF}$.

Interior Angle Axiom **4**
If P is inside $\angle EFG$, then $\angle EFP < \angle EFG$ and $\angle PFG < \angle EFG$ and $\angle EFP + \angle PFG = \angle EFG$.

Pasch's Axiom **5**
If a line passes between two vertices of a triangle and does not go through the other vertex, then it passes between it and one of the two vertices.

Triangle Inequality Theorem **(Euclid, Book I, Prop. 20, 22)** **MD 5**
Three lengths can be of triangle sides if and only if the sum of the lengths of any two sides is greater than the length of the third side.

Continuity Theorem **MD 5**

1. A line that passes through a point inside a circle intersects the circle exactly twice.
2. A circle that passes through points inside and outside a circle intersects it exactly twice.

Archimedes' Axiom **6**
Given any two segments $\overline{EF} < \overline{GH}$, there exists a natural number, n, such that $n|\overline{EF}| > |\overline{GH}|$.

Crossbar Theorem **MD 7**
A ray from a triangle vertex that is inside this angle intersects the opposite side inside of it.

Example Theorem **8**
The sum of quadrilateral diagonals exceeds the sum of either pair of opposite sides.

White Belt Instruction: Foundations

Yellow Belt Instruction: Congruence

Angle–Side–Angle (ASA) Theorem (Euclid, Book I, Prop. 26) WP MD 37
Given two angles and the included side, a triangle is fully defined.

Isosceles Triangle Theorem Converse (Euclid, Book I, Prop. 6) WP MD 37
If two angles of a triangle are equal, then their opposite sides are equal.

Problem 2.1 37

Draw a segment $5.8''$, raise perpendiculars at each endpoint and bisect the right angles to form a triangle with the angle bisectors meeting at the apex. How long are the legs in 10th of an inch? Are you sure that they are equal? Are they the same length in hyperbolic geometry?

Vertical Angles Theorem (Euclid, Book I, Prop. 15) WP MD 37
Given $\overleftrightarrow{EF}$ and G, J on opposite sides of it, G, E, J are collinear iff a pair of vertical angles is equal.

Problem 2.2 38

Given $\overline{EFGH}$, if the diagonals bisect each other, prove that $\overline{EF} = \overline{GH}$ and $\overline{FG} = \overline{HE}$.

Exterior Angle Inequality Theorem (Euclid, Book I, Prop. 16) MD 38
An exterior angle of a triangle is greater than either remote interior angle.

Exterior Angle Inequality Theorem Corollaries (Euclid, Book I, Prop. 21) 38

1. The base angles of an isosceles triangle are acute.
2. A right or obtuse triangle has two acute angles.
3. Given $\overline{EFG}$ and P inside it, $\angle EGF < \angle EPF$.

Greater Angle Theorem (Euclid, Book I, Prop. 18) MD 38
If two sides of a triangle are unequal, then their opposite angles are unequal, the shorter side opposite the smaller angle and the longer side opposite the larger angle.

Greater Side Theorem (Euclid, Book I, Prop. 19) WP MD 38
If two angles of a triangle are unequal, then their opposite sides are unequal, the smaller angle opposite the shorter side and the larger angle opposite the longer side.

Problem 2.3 38

Diameters are the greatest chords. (They try not to let it go to their heads.) Proof?

Triangle Inequality Theorem Corollaries 39

1. Any side of a triangle is greater than the difference of the other two sides.
2. Given $\overline{EFG}$ and P inside it, $\overline{EP} + \overline{PF} < \overline{EG} + \overline{GF}$.
3. The sum of the medians is greater than the semiperimeter and less than the perimeter.

Orange Belt Instruction: Parallelograms

Construction 3.2 **92**
Three highways intersect to make a triangle with sides of given lengths. The highways are connected by arcs of their excircles. Locate the exit ramps to these arcs.

Problem 3.1 **92**
Two country roads intersect at an arbitrary angle. We wish to pave an arc connecting them and going around the corner of a farmer's field, which is on the angle bisector of the two roads.

Transversal Lemma **(Euclid, Book I, Prop. 27)** **WP MD 92**
If alternate interior angles are equal, the two lines crossed by the transversal are parallel.

Transversal Theorem **(Euclid, Book I, Prop. 29)** **WP MD 93**
If the two lines crossed by a transversal are parallel, then alternate interior angles are equal.

Transversal Theorem Corollary **WP MD 93**
Two lines are parallel if and only if a perpendicular to one is perpendicular to the other.

Rectangle Bimedian Theorem **93**
A rectangle's bimedians are equal to the sides they do not cut, and their extensions are parallel.

Pairwise Parallels/Perpendiculars Theorem **WP 93**
If the rays of two angles are pairwise parallel or pairwise perpendicular, then the angles are equal; the only exception is for pairwise perpendicular angles with their vertices inside the other angle, so the angles are supplementary. (This is called quadrilateral angle sum theorem corollary #1.)

Equal Perpendiculars Theorem **93**
Perpendiculars through a point inside a square are equally cut by opposite sides of the square.

Construction 3.3 **(Euclid, Book I, Prop. 31)** **MD 93**
Construct a line parallel to a given line through a point not on the line.

Construction 3.4 **94**
Construct a line through a point that meets a given line at a given angle.

Problem 3.2 **MD 94**
Prove that, if two lines are parallel and a line cuts one of them, it also cuts the other.

Problem 3.3 **94**
Given $\overline{EFG}$, draw a line through the incenter parallel to $\overleftrightarrow{EF}$ that intersects $\overline{EG}$ and $\overline{FG}$ at J and K, respectively. Prove that $\overline{JK} = \overline{EJ} + \overline{FK}$.

Problem 3.4 **94**

Given two parallel lines, draw a transversal that cuts one line at an angle such that it is twice (or three times or five times) one of the angles that the other line is cut at.

Problem 3.5 **94**

You are given two points, a circle, and a line. Draw a circle that passes through the two points and whose common chord with the given circle is parallel to the given line.

Angle Sum Theorem **(Euclid, Book I, Prop. 32)** **WP MD 94**

Interior angles of a triangle sum to one straight angle; that is, $\alpha+\beta+\gamma=\sigma$.

Exterior Angle Theorem **(Euclid, Book I, Prop. 32)** **WP MD 95**

An exterior angle equals the sum of the remote interior angles.

Isosceles Angle Theorem **95**

If α is the apex angle of an isosceles triangle, a base angle is $\rho-\frac{1}{2}\alpha$, which is also $\frac{1}{2}(\sigma-\alpha)$. The supplement of the base angle is $\rho+\frac{1}{2}\alpha$, and double the base angle is $\sigma-\alpha$.

Quadrilateral Angle Sum Theorem **95**

Interior angles of a quadrilateral sum to two straight angles.

Quadrilateral Angle Sum Theorem Corollaries **95**

1. If opposite quadrilateral angles are right, then the other two angles are supplementary.
2. Let $\overline{EFGH}$ be tangential with incenter I. Then, $\angle EIF+\angle GIH=\sigma=\angle FIG+\angle HIE$.
3. Let $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ be tangent to an I-circle and $\overleftrightarrow{FG}$ also tangent. Then, $\angle FIG=\rho$.

Triangle Centers' Angles Theorem **95**

Let $\overline{EFG}$ have orthocenter H, incenter I and circumcenter O.

1. If $\angle E<\rho$ and $\angle F<\rho$, then $\angle EHF$ is supplementary to $\angle G$. $\angle EHF+\angle G=\sigma$
2. $\angle EIF$ is a right angle plus half of $\angle G$. $\angle EIF=\rho+\frac{\angle G}{2}$
3. If $\angle G\le\rho$, then $\angle EOF$ is double it. $\angle EOF=2\angle G$

Polygon Angle Sum Theorem **WP** **95**

1. Interior angles of n adjacent triangles sum to n straight angles.
2. Exterior angles of n adjacent triangles with a convex union sum to two straight angles.

Angle–Angle (AA) Similarity Theorem **WP** **96**

Two corresponding angles equal is sufficient to prove the similarity of two triangles.

Problem 3.6 **96**

From a house in the country, construct a dirt road to a straight paved road, the latter twice as fast as the former, to minimize travel time to a nearby town on the paved road.

Pairwise Parallel/Perpendicular Similarity Theorem **WP** **96**

If the side extensions of triangles are pairwise parallel or pairwise perpendicular, they are similar.

Problem 3.7 **96**

Given $\overline{EFG}$ with incenter I and excenter X, prove that $\overline{IGE} \sim \overline{FGX}$.

Problem 3.8 **96**

Prove that, if the bisector of an exterior angle is parallel to the opposite side, then the triangle is isosceles. Is the given angle the base or the apex angle of the isosceles triangle?

Problem 3.9 **96**

Two lines meet several centimeters off the paper. Perform these constructions:

1. Replicate the angle that they make; and 2. Bisect the angle that they make.

Problem 3.10 **96**

Design a trucker's triangular hazard reflector. Draw an equilateral triangle and then another one with the same center and orientation, but with sides half of the outer lengths.

Problem 3.11 **97**

Through a point on a circle, draw a chord twice as long as it is from the center.

Lambert Theorem **97**

Lambert quadrilaterals (three right angles) are right rectangles.

Lambert Theorem Corollary **MD 97**

A parallelogram with at least one right angle is a right rectangle.

Kite Theorem **98**

The diagonals of a kite are perpendicular, and the non-definitional diagonal is bisected.

Kite Altitudes Theorem **98**

If $\overline{EFGH}$ is a kite, $\overline{EFH} \cong \overline{GFH}$ and H_G, H_E pedal triangle vertices in $\overline{EFG}$, then $\overline{H_G F H_E H}$ is a kite.

Viviani Sum Theorem **98**

The altitude to a leg of an isosceles triangle is equal to the sum of the distances to the legs from any point on the base.

Parallelogram Diagonals Theorem **MD 104**
A quadrilateral is a parallelogram if and only if the diagonals bisect each other.

Right Triangle Median Theorem **MD 104**
The median to the hypotenuse of a right triangle is half of the hypotenuse.

Mid–Segment Theorem **MD 104**
1. A mid-segment of a triangle is half the other side, and their extensions are parallel.
2. A line parallel to the base of a triangle that bisects one side also bisects the other side.
3. Given $\overline{EFG}$, J on the same side of $\overleftrightarrow{GE}$ as F, $\overleftrightarrow{M_{GE}J} \parallel \overleftrightarrow{EF}$ and $\overline{M_{GE}J} = \frac{1}{2}\overline{EF}$, then $J \equiv M_{FG}$.

Medial Triangle Theorem I **105**
The medial triangle is congruent to the three triangles around it and all five triangles are similar.

Medial Triangle Theorem II **105**
The feet of perpendiculars dropped from a triangle's apex onto its base angle bisectors define a line that is parallel to the base.

Medial Triangle Theorem III **105**
Perpendiculars dropped on interior and exterior angle bisectors from the other vertices of a triangle have their feet on the extensions of the sides of its medial triangle.

Problem 3.17 **105**
Prove that the incenter of a triangle lies inside its medial triangle.

Construction 3.6 **105**
Construct a triangle given the legs and the median to the base.

Construction 3.7 **105**
Construct a triangle given the base, a base angle, and the median to the opposite leg.

Construction 3.8 **106**
Given $\overline{G_1G_2} = \overline{G_2G_3}$ on line l_1 and an arbitrary point E_1 on line l_2, find E_2 and E_3 so $\overline{E_1E_2} = \overline{E_2E_3}$.

Construction 3.9 (Euclid's solution) **(Euclid, Book VI, Prop. 10)** **MD 106**
Trisect a segment.

Construction 3.10 **106**
Construct a quadrilateral given the four sides and one bimedian.

Triangle Frustum Mid–Segment Theorem **MD 107**
A triangle frustum's mid-segment is the semisum of the base and the top, and parallel to them.

Orthocenter Theorem **MD 109**
The altitudes are concurrent at a point that we will call the orthocenter.

Medial Triangle Orthocenter Theorem **109**
The circumcenter of a triangle is the orthocenter of its medial triangle.

Half–Scale Orthocenter to Vertex Theorem **109**
The distance from the orthocenter to a vertex of the medial triangle is half the corresponding length in its parent triangle.

Problem 3.20 **110**
Given $\overline{EFG}$ with $\angle E = \rho$, let E' be the foot of the perpendicular dropped on $\overline{FG}$ and F_{EG} be the reflection of F. Prove that $\overleftrightarrow{F_{EG}E'} \perp \overleftrightarrow{GM_{EE'}}$.

Problem 3.21 **110**
Given $\overline{EFGH}$ a rectangle and F' the foot of the perpendicular dropped on $\overline{EG}$, prove that $\angle M_{GH}M_{EF'}F = \rho$.

Problem 3.22 **110**
Given $\overline{EFG}$, build squares on the exteriors of $\overline{EG}$ and $\overline{FG}$ with sides $\overline{EE''}$ and $\overline{FF''}$, respectively. Prove that $P := \overline{EF''} \cap \overline{FE''}$ is on the altitude $\overline{GG'}$.

Construction 3.11 (Three Modern Solutions) **111**
Trisect a segment.

Isosceles Triangle Frustum Theorem **MD 112**
In an isosceles triangle frustum: (1) base angles are equal; (2) opposite angles are supplementary; (3) legs are equal; (4) diagonals are equal; and (5) the frustum is cyclic. And the converses.

Triangle Frustum Theorem I **112**
Given $\overline{EF}$ with midpoint M_{EF} and E', M'_{EF}, F' the feet of perpendiculars dropped on a line that does not intersect $\overline{EF}$, then $2\overline{M_{EF}M'_{EF}} = \overline{EE'} + \overline{FF'}$.

Triangle Frustum Theorem II **112**
Let E, F, G be collinear, M_{FG} the midpoint of $\overline{FG}$, $2\overline{EF} = \overline{FG}$ and E', F', M'_{FG}, G' be the feet of perpendiculars dropped on a line that does not intersect $\overline{EG}$. Then, $3\overline{FF'} = 2\overline{EE'} + \overline{GG'}$.

Problem 3.23 **113**
Let $\overline{EFG}$ be a right triangle with $\angle EFG$ right and F' the foot of the altitude to the hypotenuse. From F' drop perpendiculars onto $\overline{EF}$ and $\overline{FG}$ with feet F'_G and F'_E, respectively. From F'_G and F'_E drop perpendiculars onto $\overline{EG}$ with feet F''_G and F''_E, respectively. Prove that (1) $\overline{F'F''_G} = \overline{F'F''_E}$; and (2) $\overline{FF'} = \overline{F'_GF''_G} + \overline{F'_EF''_E}$.

Problem 3.24 **113**

Given $\overline{EFG}$ with midpoints M_{EF}, M_{FG}, M_{GE} and medial point C, let $E', F', G', M'_{EF}, M'_{FG}, M'_{GE}, C'$ be the feet of perpendiculars dropped on a line external to $\overline{EFG}$, respectively; prove that $\overline{EE'} + \overline{FF'} + \overline{GG'} = \overline{M_{EF}M'_{EF}} + \overline{M_{FG}M'_{FG}} + \overline{M_{GE}M'_{GE}} = 3\overline{CC'}$

Problem 3.25 **113**

Given $\overline{EFGH}$ a parallelogram and E', F', G', H' the feet of perpendiculars dropped onto a line exterior to the parallelogram, prove that $\overline{EE'} + \overline{GG'} = \overline{FF'} + \overline{HH'}$.

Problem 3.26 **113**

Given $\overline{EFG}$ and E', F', G', M'_{EF} the feet of perpendiculars dropped from E, F, G, M_{EF} onto a line through the medial point C that does not cut $\overline{EF}$; prove $\overline{GG'} = \overline{EE'} + \overline{FF'}$.

Parallel Lines Theorem **MD 114**

Two lines never intersect if and only if they are everywhere equidistant.

Construction 3.12 **114**

Construct the two external tangents to two circles of different radii.

Construction 3.13 **114**

Construct the two internal tangents to two disjoint circles of different radii.

Medial Parallelogram Theorem I **MD 115**

Connecting the midpoints of consecutive sides in a quadrilateral form a parallelogram.

Medial Parallelogram Diagonals Theorem **115**

1. Medial parallelogram side extensions are parallel to a diagonal of the parent quadrilateral.
2. The perimeter of the medial parallelogram equals the sum of the parent diagonals.

Medial Parallelogram Theorem II **115**

Given $\overline{EFGH}$ not a parallelogram or a triangle frustum, then $M_{FG}, M_{FH}, M_{HE}, M_{EG}$ are the vertices of a parallelogram, as are $M_{EF}, M_{EG}, M_{GH}, M_{FH}$. (The order depends on the shape of $\overline{EFGH}$.)

Varignon Theorem **115**

The bimedians of a quadrilateral bisect each other.

Right Triangle Theorem **(Euclid, Book VI, Prop. 8)** **WP** **116**

The altitude to the hypotenuse of a right triangle forms two triangles similar to it and each other.

Nested Triangle Theorem **WP** **116**

Nested triangles (two transversals that intersect outside two parallel lines) are similar.

Intersecting Chords Theorem (without proof) (Euclid, Book III, Prop. 35) WP 135
If two chords of a circle intersect inside the circle, the product of the two segments of one is equal to the product of the two segments of the other.

Intersecting Secants Theorem (without proof) (Euclid, Book III, Prop. 36, 37) MD 135
If two secants of a circle intersect outside the circle, the product of the two segments of one, from the intersection to where the circle cuts it, is equal to the product of the two segments of the other, from the intersection to where the circle cuts it.

Altitude and Diameter Theorem (without proof) WP 135
The product of two sides of a triangle is equal to the product of the altitude to the third side and the diameter of the circumcircle.

Triangle Similarity Theorem (without proof) (Euclid, Book VI, Prop. 4, 5) WP MD 135
If two triangles are similar, their corresponding sides are proportional.

Side–Splitter Theorem (without proof) (Euclid, Book VI, Prop. 2) WP MD 135
A line through two sides of a triangle parallel to the third side divides those sides proportionally.

Problem 3.59 135
Any line through two circles' touching point is cut in proportion to their diameters.

First Law of Sines WP 136
$\frac{e}{\sin\alpha} = \frac{f}{\sin\beta} = \frac{g}{\sin\gamma} = 2R$

Second Law of Sines 136
$\frac{e-f}{e+f} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$

First Law of Cosines WP 136
$g^2 = e^2 + f^2 - 2ef\cos\gamma$

Second Law of Cosines 137
$g = e\cos\beta + f\cos\alpha$

Surveying Techniques to Measure or Lay Off Lengths 138

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Green Belt Instruction: Triangle Construction

$\frac{e-f}{e+f}=\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$

$\frac{e}{f}=\frac{1+T}{1-T}$ with $T=\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$

Let $\overline{EFGH}$ be tangential but not a square with I its incenter, $I_{EF}, I_{FG}, I_{GH}, I_{HE}$ the incircle's touching points and P the intersection of $\overline{I_{EF}I_{GH}}$ and $\overline{I_{FG}I_{HE}}$. Draw a line through P perpendicular to $\overline{IP}$ and label its intersections with $\overline{HE}$ and $\overline{FG}$ as J and K, respectively. Prove $\overline{I_{HE}J}=\overline{I_{FG}K}$.

Let $\overline{EFGH}$ be tangential with I its incenter. Draw lines through E,F,G,H perpendicular to $\overline{IE},\overline{IF},\overline{IG},\overline{IH}$, respectively. Let J,K,L,M be the intersections of these lines that are long of $\angle EIF, \angle FIG, \angle GIH, \angle HIE$, respectively. Prove that I is the bi-medial of $\overline{JKLM}$.

Given cyclic quadrilateral $\overline{EFGH}$ such that $\angle E \neq \angle G$, let I and J be the incenters of $\overline{EGH}$ and $\overline{EFG}$, respectively. Prove that $\overline{EFGH}$ is tangential if and only if $\overline{IJFH}$ is cyclic.

Given two overlapping circles, they are orthogonal if and only if any of these conditions hold:

1. Radii of the two circles to an intersection point are perpendicular.
2. A radius of one circle to an intersection point is tangent to the other circle.
3. The circle whose diameter is from center to center passes through their intersections.

Given a circle and a point, construct an orthogonal circle through the point.

Problem 4.42 **216**
Given three circles with three touching points, prove that the circle through the three touching points is orthogonal to all three given circles.

Problem 4.43 **216**
Given two orthogonal circles, prove that the two lines from their two intersections to a point on one circle meet the other circle at diametrically opposite points.

Problem 4.44 **216**
Given $\overline{EFGH}$ cyclic with $\overline{EF}$ a diameter and T the bi-medial point, prove that a circle with common chord $\overline{GH}$ is orthogonal if and only if it passes through T.

Problem 4.45 **216**
Given $\overline{EF}$, a diameter of ω, and G any point on ω, prove that the circles through E, M_{EF}, G and through F, M_{EF}, G are orthogonal.

Long Circle Theorem Corollary **216**
Given $\overline{EFG}$ with circumcircle ω, then E, F and I are equidistant from the long center, L_G.

Problem 4.46 **216**
Tangents to a circle at E, F meet at G; prove that the $\overline{EFG}$ incenter is on the circle.

Problem 4.47 **216**
Given $\overline{EFG}$ with incenter I and excenters X, Y, Z, prove that the circles with diameters $\overline{IX}$ and $\overline{YZ}$ are orthogonal.

Problem 4.48 **216**
Given $\overline{EFG}$ with orthocenter H, prove that the circles with diameters $\overline{EH}$ and $\overline{FG}$ are orthogonal.

Orthogonal Lens Area Theorem **216**
The overlap of orthogonal circles with radii R and r has area $A = (R^2 - r^2)\,\text{atan}\frac{r}{R} + \frac{\pi}{2}r^2 - Rr$ or $r^2\left(\frac{\pi}{2} - 1\right)$ if $R = r$. For general but equal circles, $A = r^2(\theta - \sin\theta)$ for θ not necessarily $\frac{\pi}{2}$.

Red Belt Instruction: Famous Theorems

Franzsen's Theorem (without proof) **240**

Let d be the distance from the incenter to the Euler segment, s the semiperimeter, μ (Greek: mu) the longest side and ν (Greek: nu) the longest median. Then, $\frac{d}{s} < \frac{d}{\mu} < \frac{d}{\nu} < \frac{1}{3}$.

Orthocenter and Wallace Line Theorem **241**

The Wallace line determined by P bisects $\overline{PH}$ if H is the orthocenter.

Euler Circle Lemma **241**

Given $\overline{EFG}$ with E', F', G' the feet of the altitudes, then $E', F', G', M_{EF}, M_{FG}, M_{GE}$ are concyclic.

Euler Circle Theorem **241**

Given $\overline{EFG}$ with E', F', G' the feet of the altitudes and H the orthocenter, the following nine points are concyclic: E', F', G' and M_{EF}, M_{FG}, M_{GE} and M_{EH}, M_{FH}, M_{GH}.

Euler Center Theorem **242**

The center of the Euler circle is the midpoint of the Euler segment.

Euler Radius Theorem **242**

The radius of a triangle's Euler circle is half its circumradius.

Euler Diameter Theorem **242**

$\overline{M_{EH}M_{FH}M_{GH}} \cong \overline{M_{FG}M_{GE}M_{EF}}$ and $\overline{M_{EH}M_{FG}}, \overline{M_{FH}M_{GE}}, \overline{M_{GH}M_{EF}}$ are diameters of the Euler circle.

Euler Bisection Theorem **242**

The Euler circle bisects any segment from the orthocenter to the circumcircle.

Problem 5.2 **242**

Prove that the circumcircle of a triangle is the Euler circle of a triangle whose vertices are the given triangle's incenter and two of its excenters.

Problem 5.3 **243**

Given $\overline{EFG}$, prove that O, M_{EF}, F, M_{FG} are concyclic and that this circle is congruent to the Euler circle of $\overline{EFG}$.

Double–Long Triangle Theorem I **243**

$\overline{EFG}$ is the orthic triangle of its double-long triangle, $\overline{XYZ}$.

Double–Long Triangle Theorem II **243**

The double-long triangle is a homothetic double of the long triangle.

Double–Long Triangle Theorem III **243**

The circumcircle of a triangle is the Euler circle of its double-long triangle.

Problem 5.8 **248**
Let ω_1 and ω_2 with centers O_1 and O_2 have common point E. Let F and G be points on ω_2. Also, let $F'' := \omega_1 \cap \overleftrightarrow{EF}$ and $G'' := \omega_1 \cap \overleftrightarrow{EG}$. Prove $\overleftrightarrow{FG} \parallel \overleftrightarrow{F''G''}$.

Lemma 5.8 **248**
Given the base and the orthocenter, if it is not on the base, a triangle is fully defined.

Problem 5.9 **248**
Let ω_1 and ω_2 with centers O_1 and O_2 have common chord $\overline{EF}$. *Let* $J := \overleftrightarrow{O_1F} \cap \omega_1$ and $M := \overleftrightarrow{O_1F} \cap \omega_2$ and $K := \overleftrightarrow{O_2F} \cap \omega_1$ and $L := \overleftrightarrow{O_2F} \cap \omega_2$. (Assuming J, M, K, L exist.) Prove:

1. $\overrightarrow{JK}, \overrightarrow{EF}, \overrightarrow{LM}$ are concurrent at a point P.
2. J, E, M, P are concyclic.
3. L, E, K, P are concyclic.

Problem 5.10 **249**
Prove that the orthocenter of a triangle is the incenter of the triangle whose vertices are where the given triangle's altitudes cut its circumcircle.

Problem 5.11 **249**
Prove that the orthic triangle and the tangential triangle are homothetic and that their homothetic center is on the Euler line, but that it is not the orthocenter.

Problem 5.12 **250**
Prove that the parent triangle is antipedal if the orthocenter is the pedal point.

Problem 5.13 **250**
Prove that the double-long triangle is antipedal if the incenter is the pedal point.

Problem 5.14 **250**
Given $\overline{EFG}$ with E, F, G counterclockwise, find:

1. P such that $\angle PEF = \angle PFG = \angle PGE$. Call this angle α; P is the first Brocard point.
2. Q such that $\angle QEG = \angle QFE = \angle QGF$. Call this angle β; Q is the second Brocard point.
3. Prove that $\alpha = \beta$. This is called the Brocard angle.

Isometric Transformations without Linear Algebra **254**

Problem 5.30 **255**
Construct a nuclear power plant in an area where there are three parallel highways, so it is at the center of an equilateral triangle with a vertex on each highway, where military bases will be built.

Right Triangle Theorem Corollaries **(Euclid, Book VI, Prop. 13)** **300**

1. The altitude is the geometric mean of the projections; $h = \sqrt{u'v'}$.
2. Each leg is the geometric mean of the leg's projection and the hypotenuse; $u = \sqrt{u'w}$.
3. The product of the altitude and the hypotenuse is the product of the legs; $hw = uv$.

Incenter Ratio Theorem **301**

The incenter cuts the bisector of an angle as the sum of its adjacent sides is to its opposite side.

Cheat Sheet!!!

I know that many of you are like, "Just show me the memorization sheet and I'll memorize it!" You will never excel at geometry with that attitude, but the theorems here will at least allow you to stumble through beginning geometry with a "C." And you do not have to memorize it – *Geometry–Do* exams are all open-book. Also, dog-ear the glossary so you can open to it quickly.

If you cannot learn these few theorems, then buy a second-hand six-string and be a juke box hero, like *Foreigner* said you could be. I hear the NBA is hiring! Lots of financial opportunities in round ball! I am not here just for the gifted students – I am also trying to help the bums!

Triangle Inequality Theorem **5**
Three lengths can be of triangle sides if and only if the sum of the lengths of any two sides is greater than the length of the third side.

Side–Angle–Side (SAS) Theorem **15**
Given two sides and the angle between them, a triangle is fully defined.

Isosceles Triangle Theorem **15**
If two sides of a triangle are equal, then their opposite angles are equal.

Side–Side–Side (SSS) Theorem **16**
Given three sides that satisfy the triangle inequality theorem, a triangle is fully defined.

Center Line Theorem **18**
An angle bisector and a perpendicular bisector coincide if and only if the triangle is isosceles.

Mediator Theorem **19**
A point is on the perpendicular bisector iff it is equidistant from the endpoints of the segment.

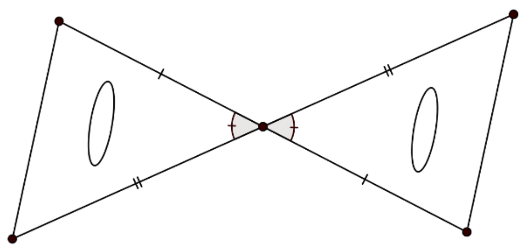
Angle–Side–Angle (ASA) Theorem **37**
Given two angles and the included side, a triangle is fully defined.

Isosceles Triangle Theorem Converse **37**
If two angles of a triangle are equal, then their opposite sides are equal.

Vertical Angles Theorem **37**
Given $\overleftrightarrow{EF}$ and G, J on opposite sides of it, G, E, J are collinear iff a pair of vertical angles is equal.

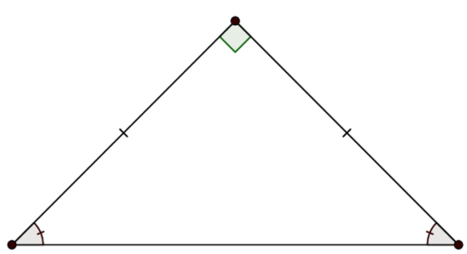
Exterior Angle Inequality Theorem **38**
An exterior angle of a triangle is greater than either remote interior angle.

The first two pages are first-year geometry, and the third page is second-year geometry. These are not the best or most interesting theorems; these are the ones most likely to be cited in proofs. SAS congruence is by far the most often cited theorem in geometry. You should always be on the lookout for two triangles with two sides and the included angle in common. As soon as you see that, just say "by SAS" and all the corresponding sides and angles are pairwise equal.



Do not label all six of these equalities – that would clutter your figure – draw a rough oval shape inside the two triangles to indicate their congruence. If you have two pairs of congruent triangles, then draw one pair with fat ovals and the other pair with skinny ovals. This notation does not always work, especially if the triangles overlap, but it is good practice with most figures.

Vertical angles are ubiquitous, but do not clutter your figure by labeling them all; just keep their equalities in mind. Sides are labeled with lowercase letters, e, f, g, and angles with Greek letters, α, β, γ, but it is usually best not to use these labels. Label two sides equal with little hash marks and label two angles equal with angle symbols with hash marks through them. If you have a second pair of equal sides or equal angles in the same figure, use double hash marks. It is their equality that you need, not their names. This is especially clarifying for things that are equal by transitivity. Labeling segments e, f, g and then writing somewhere that $e = f$ and somewhere else that $f = g$ does not clarify that $e = g$. Always label right angles with a square angle symbol.



Be on the lookout for isosceles triangles. As soon as you see a triangle with two sides equal, immediately label the two base angles equal; or, if you see two angles in a triangle equal, label the two legs equal. Even if you do not yet know where you are going with this, assume that the triangle would not be given to be isosceles if these equalities were not important.

The polygon angle sum theorem is included on this list because it often appears on *Common Core* exams. If asked for the interior angle of a regular polygon, subtract the exterior angle from $180°$. The only other *Common Core* theorems are the vertical angles and angle sum theorems. (Sometimes they use the intersecting chords theorem to create quadratic equations.) Some angles will be linear functions of x. The vertical angles theorem allows you to set them equal; the angle sum theorem allows you to set their sum equal to $180°$. Don't ask what x represents, it is there only to set up an algebra exercise. *Common Core* "geometry" is just remedial algebra!

Principal Results of *Geometry–Do* in Alphabetical Order

Index of Names

Boldface names are geometers. Boldface pages have theorems named after the geometer.

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[126] Dover bound Dubnov with Fetisov [1954] 1963. This first edition of *Proof in Geometry* initiated the Pitot theorem converse controversy. But Fetisov's [1977] 1978 Mir edition is revised and expanded and worth buying separately.
[127] Glagolev died in 1945; this 3rd edition that Fetisov criticized is posthumous, which probably explains why Fetisov did not name Glagolev, though he quotes him verbatim.

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[128] Givental does not say he is doing an abridgment, but the Russian text of Kiselev's *Planimetry* has 302 sections and Givental has only 260. Also, Givental is a faithless translator. Равенство means equality, not congruence. Consider this: "**11. Равенство конечных прямых.** Два отрезка прямой считаются равными, если они при наложении совмещаются." This should be, "**11. Equality of finite lines.** Two line segments are considered equal if superposition makes them coincide." For Givental, this is section six because he is omitting sections. He translates it, "**6. Congruent and non-congruent segments.** *Two segments are congruent if they can be laid one onto the other so that their endpoints coincide.*" Modern American textbooks use "congruent" to the complete exclusion of "equal," but this does not give the translator of an historical document the right to make this change unannounced. Also, note that Givental talks around "superposition" because, while American textbooks rely on it, it is too big of a word. Givental is trying to make this 19th century Russian sound like an early founder of *Common Core* geometry. He is not. While I do not agree with Kiselev on everything – specifically, I do not rely on superposition – *Planimetry* is a useful historical document. I cannot recommend this faithless translation. Read the original Russian if you can.

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Glossary

If a student looks up "rectangle" in the index of *Glencoe Geometry*, he finds that the word appears on pages 24, 58 and 423–429. Pages 23 and 58 are no help because *Glencoe* assumes students already knows what a rectangle is, and they just state the formulas for perimeter and area. Note the complete lack of foundations; *Glencoe* is multiplying base times height without having ever defined multiplication and division of lengths. Suppose I construct an isosceles right triangle and then attempt to divide the hypotenuse by the leg. Is this quotient a length? We will never find out by reading *Glencoe!* Also, note that it took over four hundred pages to get around to a formal definition of a rectangle. What were they doing in the meantime? Opening the book at random to page 93, we find a photo of a man cutting a girl's hair and learn that stylists must attend cosmetology school and obtain a license. Colorful charts and graphs display how many customers a salon is seeing on the weekends over a six-month period. *Glencoe* concludes, "Survey data supports a conjecture that the amount of business on the weekends has increased, so the owner should schedule more stylists to work on those days." *Ahem!* Getting back to rectangles, let us see what *Glencoe Geometry* writes on page 423:

> *By definition, a rectangle has the following properties.*
> - *All four angles are right angles.*
> - *Opposite sides are parallel and congruent.*
> - *Consecutive angles are supplementary.*
> - *Diagonals bisect each other.*

By definition??? Quack! These are *easy* theorems that any 12-year-old in Russia could prove. But debunking mountebanks is not my job. The textbooks that I recommend are Casey's *Redux*[129], Hall & Stevens' *A School Geometry* and Wentworth's *Plane Geometry*.

Students beware! A common cause of falling behind one's mates is that you made no effort to learn the terminology before a lecture and then spent much of the lecture with your head down thumbing through your textbook looking for definitions. A common cause of the entire class failing the final exam is that they had an accommodating teacher who spent half of every lecture defining words and consequently did not cover all the material needed to take the final exam.

The *Note to Teachers* states, "The terms in the glossary are color coded to the chapters where they are introduced. New terms are in **boldface**, but I do not pause to define them. It is your job to tell students at the end of each day which terms to look up in the glossary for the next lecture." If your teacher is not following these instructions, then ask him or her to do so!

[129] Daniel Callahan translated and edited a 19th century textbook by the Irish mathematician John Casey.

Adjacent	Two disjoint triangles with a common side (common for its full length), or two angles with a common ray (common vertex and direction)
Altitude	The perpendicular from a triangle vertex to the opposite side's extension
Analytic	Knowledge contained in the given information
Angle	Two rays, called the sides, sharing a common endpoint, called the vertex. $\angle F$ if there is one angle at F or $\angle EFG$ for the angle between $\overrightarrow{FE}$ and $\overrightarrow{FG}$.

Acute	An angle that is less than a right angle
Alternate Interior	Angles on opposite sides of a transversal and between the two given lines
Apex	The angle opposite the base of a triangle
Base	In a triangle with a base, the angles at either end
Central	An angle whose vertex is the center of a circle
Complementary	Two angles that sum to one right angle
Conjugate	Angles that sum to two straight angles
Consecutive	Angles both interior (exterior) on the ends of a side
Elevation	One ray is on level ground and the other is above it
Exterior	The angle supplementary to an interior angle
Inscribed	An angle inside a circle with its vertex on the circle
Interior	An angle inside a triangle or quadrilateral at a vertex
Obtuse	An angle greater than right and less than straight
Parallelism	In hyperbolic geometry; $2\,\mathrm{atan}(e^{-x})$ with x height
Right	The bisection of a straight angle
Skew	The difference of the base angles of a triangle
Straight	An angle whose rays are collinear and opposed
Supplementary	Two angles that sum to one straight angle
Vertical	Angles across from each other at an intersection

Anticenter	The point of concurrency of the maltitudes; it exists for cyclic quadrilaterals
Apex	The triangle vertex opposite the base
Arc	Part of a circle; within equal circles, angles at the center and the arcs they cut off are a transformation of each other.
Area	The measure of the size of a triangle or a union of disjoint triangles
Auxiliary	Lines or arcs not given whose intersection goes beyond analytic
Axiom	A proposition that is assumed without proof for the sake of studying the consequences that follow from it
Base	The side of an isosceles triangle bracketed by the equal angles The side of a triangle designated as such, or the one that it is built on
Between	1. If F is between E and G, then F is also between G and E and there exists a line containing the points E, F, G. (Between implies that the three points are distinct.)

2. If E and G are two points on a line, then there exists at least one point F lying between E and G and at least one point H such that G lies between E and H.
3. Of any three collinear points, there is exactly one between the other two.

Bi–Conditional	A statement of the form p if and only if q. It is true if both p and q are true or both p and q are false. p implies q; also, q implies p. Proof of neither implication can cite the other implication. If and only if is abbreviated iff.	
Bi–Medial	The intersection of the diagonals of a quadrilateral	
Bimedian	A segment joining the midpoints of opposite sides of a quadrilateral	
Bisect $\frac{1}{2}$	To divide a segment or an angle into two equal parts, called halves	
Center Line	The mediator of the base of an isosceles triangle or a semicircle	
Centroid	The balance point of a uniform plate. Proving that a triangle's medial point is its centroid requires calculus, so these terms are *not* interchangeable.	
Chord	The segment between two points on a circle	
	Common	The segment between the intersection points of two circles
Circle	All the points equidistant from a point, which is called the center	
Circum	**circle**	A circle that intersects a figure at its vertices
	center	The center of the circumcircle
	radius	The radius of the circumcircle; R
Closing the Horizon	All the angles around a point must sum to 2σ.	
Collinear	A set of points that are all on the same line	
Concentric	Two or more circles with the same center but different radii	
Concurrent	Three or more lines or arcs that intersect at the same point	
Concyclic	Four or more points on the same circle	
Condition	Constraints that a figure either conforms to or not	
Conformal	A transformation that preserves angles; e.g., scaling and circle inversion	
Congruent $\cong$	Two triangles whose areas and whose sides and whose interior angles are equal	
Contradiction, Proof by	To prove that statement p implies statement q, assume that p is true and q is not true and show that this is impossible.	
Converse	Given the statement that p implies q, the statement that q implies p	

Convex	Any segment between two points interior to two sides is inside the figure	
Defect	In hyperbolic geometry; $\sigma - (\alpha + \beta + \gamma)$ for a triangle of a given size	
Diagonal	Segments connecting non-consecutive quadrilateral vertices	
	Definitional	The adjacent side of the two triangles in a quadrilateral
Diameter	A chord that crosses the center of a circle	
	Diametrically Opposed	The endpoints of a diameter
Dichotomy	Proof by contradiction when there are two alternatives	
Discussion	The necessary and sufficient conditions for a solution, and how many solutions	
Disjoint	Figures that do not overlap; their areas form an additive group (This includes touching circles and adjacent triangles, if outside each other.)	
Disjoint	There is zero probability of any points being inside both figures.	
Edubabble	Ridiculously fluffy words and silly sloganeering intended to obfuscate and confuse (American high-school teachers do not study their subject in college – they will pick that up from their students' textbook – they get education babble instead.)	
Endpoint	A point at the end of a segment, arc, or ray	
Equal $=$	Comparable magnitudes that are not less than nor greater than each other	
Equidistant	Two pairs of points that define two segments of equal length	
	Lines	Any two perpendiculars between them are of equal length
Equivalence	**Class**	A set of objects that are equal, congruent, similar, or parallel
	Relation	A set and a reflexive, symmetric and transitive relation
Equivalent	Conditions, any two of which are bi-conditional	
Euler	**Center**	The center of the Euler circle
	Circle	Of a triangle, the circle through the side midpoints, altitude feet, and midpoints from the orthocenter to the vertices. This is the nine-point circle in some textbooks.
	Segment	The segment from the orthocenter to the circumcenter
Ex	**circle**	A circle tangent to a side of a triangle and to the extensions of the adjacent sides
	center	A center of an excircle
	foot	Where an exterior angle bisector cuts the extension of the opposite side; $E^{\times}, F^{\times}, G^{\times}$
	radii	The radii of the excircles; r_X, r_Y, r_Z are of $\omega_X, \omega_Y, \omega_Z$

Extend	Given $\overline{EF}$, construct $\overline{EG}$ such that F is inside $\overline{EG}$ or E is inside $\overline{FG}$.	
Field of Fire	All the points interior to the top traverse of a machine gun	
Figure	A set of points. They may be alone or joined in lines, segments, and arcs.	
Foot	The intersection when one drops a perpendicular from a point to a line	
Frustum, Triangle	The part of a triangle between the base and a cut parallel to the base	
Fully Defined	A figure with the given characteristics exists, and it is unique.	
Geometric Mean	If $\frac{a}{b} = \frac{b}{c}$ for real numbers a, b, c, then b is the geometric mean of a and c	
Half–Scale	A triangle whose sides are half the corresponding sides in another triangle	
Harmonic Division	To cut a segment $\overline{EF}$ internally and externally in the same ratio; $\frac{\overline{EG^*}}{\overline{FG^*}} = \frac{\overline{EG^\times}}{\overline{FG^\times}}$	
Homothetic	**Center**	$\overleftrightarrow{E''E} \cap \overleftrightarrow{F''F} \cap \overleftrightarrow{G''G}$ for $\overline{E''F''G''}$ homothetically double $\overline{EFG}$
	Double	A triangle whose sides are twice the lengths of another triangle's sides, and whose side extensions are pairwise parallel to that triangle's side extensions
	Triple	Analogous to a homothetic double, but of triple length
Hypotenuse	The side of a right triangle opposite the right angle	
In	**circle**	A circle that touches each side of a figure
	center	The center of the incircle
	diameter	The diameter of the incircle; d
	foot	Where an angle bisector cuts the opposite side; E^*, F^*, G^*
	radius	The radius of the incircle; r
Inside	**Segment**	A member of the set of segment points, but not an endpoint
	Figure	A point such that any line through it intersects the figure at exactly two points and the point is between them
	Triangles	A triangle whose every point is inside of or on a side of another triangle, but the triangles do not coincide
Isometric	A transformation that preserves lengths; by SSS, it also preserves angles	
Kill	**Chord**	A segment from one side of an angle to the other
	Circle	The circle of largest radius touching or inside two angles
Legs	**Triangle**	The sides other than the base or the hypotenuse
	Triangle Frustum	The sides that are not parallel

Lemma	A theorem used for proving other more important theorems	
Length	The measure of the size of a segment; the distance between its endpoints	
Line	A segment extended past both endpoints; denoted $\overleftrightarrow{EF}$ if $\overline{EF}$ is the segment	
Line of Centers	The line that passes through the centers of two circles	
Lines, Supplementary	If alternate interior angles are supplementary, the two lines crossed by the transversal are supplementary relative to that transversal.	
Locus	All the points that satisfy a condition; the plural is loci (lō’ sī)	
Long	Inside an interior angle of a triangle, but outside the triangle Inside a machine gun’s field of fire, but past its kill chord	
	Center	The intersection of an angle bisector and the circumcircle
	Circle	Around a long center through the incenter and excenter
Magnitude	A set with both an equivalence relation, $=$, and a total ordering, $\leq$	
Maltitude	Midpoint-altitude; the perpendicular dropped from the midpoint of a side of a quadrilateral onto the opposite side	
Measure	The size of sets; counts of discrete points, lengths of segments, or areas of triangles or unions of disjoint triangles.	
Medial Point	The point where the medians or the bimedians are concurrent	
Median	A segment from a vertex of a triangle to the midpoint of the opposite side	
Mediator	The perpendicular bisector of a segment	
Midpoint	The point where a segment is bisected	
Mirror Property	Light bouncing off a mirror to a point goes as far as it would to its reflection	
Mid–Segment	**Triangle**	A segment connecting the midpoints of two sides
	Triangle Frustum	A segment connecting the midpoints of the legs
Miquel	**Point**	The point defined by the Miquel theorem
	Circles	The circumcircles defined by the Miquel theorem
Napoleon Point	**First**	The center of the equilateral triangle defined in the Napoleon theorem for external equilateral triangles
	Second	The same, but for the internal equilateral triangles
Neutral Geometry	A postulate set that does not mention parallel lines; absolute geometry	
Non–Euclidean	A postulate set that contains one that contradicts the parallel postulate	

Opposite	**In a Triangle**	An angle and a side across from each other
	Of a Line	Endpoints of a segment cut by the line
	In a Quadrilateral	Two sides or two angles across from each other

Ordering, Total A set and a relation, $\leq$, that is not symmetric, but is reflexive, anti-symmetric and transitive

Orthic Reflection The reflection of the orthocenter around a side of the triangle

Orthocenter The point where the altitudes of a triangle are concurrent

Orthogonal Two arcs such that the tangents at their intersection are perpendicular

Parallel Two lines that do not intersect

Pedal Point A point from which perpendiculars are dropped onto the sides or the extensions of the sides of either a triangle or a quadrilateral

Pencil A triangle with the base infoot and the base exfoot connected to the apex (Pencil is a big word in advanced geometry, but we are not there yet. Just drawing this figure and calling it pencil is enough for high-school students.)

Perimeter The sum of the lengths of the sides of a triangle or quadrilateral

Perpendicular A line whose intersection with another line makes a right angle

Polygon The union of multiple triangles adjacent on their sides such that it is convex

Postulate The axioms that are specific to geometry, not to other branches of math

Power of the Point For a point P and a circle with center O and radius r; if $z = |\overline{OP}|$, then the power of the point is $|P| = |r^2 - z^2|$. If P is x and y distant from the circle on a chord or a secant, then $xy = |P|$.

Probability The ratio of the measure of a subset to the measure of the whole set

Projection In $\overline{EFG}$, the projection of $\overline{EG}$ and $\overline{FG}$ onto $\overline{EF}$ is $\overline{EG'}$ and $\overline{FG'}$, respectively

Quadrature Theorems proving equality of the areas of triangles or unions of triangles

Quadrilateral The union of two triangles adjacent on a side such that it is convex; $\overline{EFGH}$

	Bi–Centric	A quadrilateral that is both cyclic and tangential
	Contact	The touching points of a tangential quadrilateral's incircle connected to each other consecutively
	Cyclic	A quadrilateral for which a circumcircle exists
	Isosceles Kite	A kite with one pair of equal sides equal the diagonal
	Kite	The union of two congruent triangles; the uncommon sides that are equal are also consecutive.
	Lambert	A quadrilateral with three right angles

	Long	The quadrilateral whose vertices are long centers
	Medial Parallelogram	The midpoints of consecutive sides of a quadrilateral connected
	Orthodiagonal	A quadrilateral whose diagonals are perpendicular
	Parallelogram	The union of two congruent triangles; the uncommon sides that are equal are also opposite.
	Parent	The quadrilateral around a medial parallelogram
	Pedal	The connected feet of perpendiculars dropped from the pedal point
	Rectangle	A quadrilateral with equal angles
	Rhombus	A quadrilateral with all equal sides; plural, rhombi
	Right Kite	A kite whose two congruent triangles are right
	Right Rectangle	A rectangle with right angles
	Right Square	A right rectangle with equal sides
	Saccheri	A quadrilateral with two opposite sides equal and perpendicular to the base
	Square	A rectangle with equal sides
	Tangential	A quadrilateral for which an incircle exists
Radius	A segment from the center of a circle to the circle; plural, radii	
Random	The points in a segment or inside a triangle or circle are uniformly distributed	
Ray	A segment extended in one direction; denoted $\overrightarrow{EF}$ if $\overline{EF}$ is the segment	
Reflection	From a point, draw a line through a point and extend it an equal distance From a point, reflect it across the foot of a perpendicular dropped on a line From a line, reflect two points across a point and draw a line through them From a circle, reflect its center across a point and draw an equal circle	
Reflexive Relation	A binary relationship over a set such that every element is related to itself	
Relation	A true/false operator on an ordered pair of elements from a given set	
Secant	A line that intersects a circle at exactly two points	
Segment	All the points along the shortest path between two points; $\overline{EF}$	
Semi	Difference	Half the difference of two lengths or of two angles
	Perimeter	Half the perimeter of a triangle
	Sum	Half the sum of two lengths or of two angles
Side	Triangle	One of the three segments that form a triangle
	Quadrilateral	An uncommon segment of one of its triangles
	Consecutive	Quadrilateral sides that share an endpoint
Similar $\sim$	Two triangles with all corresponding angles equal	
Steiner Line	A line parallel to the Wallace line such that the Wallace line is halfway between it and the Wallace line's pedal point; also called the ortholine or orthocentric line.	

Subtend	A chord creating an equivalence class of inscribed angles to one side of it.
Subtend at Center	A chord creating an angle with its vertex at the circle center.
Summit	The side of a Saccheri quadrilateral that is opposite the base
Symmetric Relation	A relation that can be stated of two things in either order
Synthetic	Knowledge that remains after the auxiliary lines and arcs are erased
T & V	The transversal and vertical angles theorems, used in some combination
Tangent	A line that touches a circle; if length is mentioned, this means the length of the segment between the touching point and the point that defines the tangent line.
	Cut — The segment of an internal tangent that is between the external tangents
	External — A line tangent to two circles that does not go between their centers, or the segment between touching points
	Internal — A line tangent to two disjoint circles that goes between their centers, or the segment between touching points
Theorem	A statement requiring proof using postulates or already proven theorems
Torricelli	These terms apply only to triangles with angles less than 2φ.
	Apex — The apex of an equilateral triangle built on the exterior of a side of a triangle; given $\overline{EFG}$, E'' is across from E.
	2nd Apex — The apex of an equilateral triangle built on the interior of a side of a triangle; in this book, it is not labeled.
	Point — The point of concurrency of the Torricelli segments; U
	2nd Point — The point of concurrency of the 2nd Torricelli segments; V
	Segment — Connect a vertex of a triangle to the Torricelli apex across from it; *e.g.*, $\overline{EE''}$, $\overline{FF''}$, $\overline{GG''}$
	2nd Segment — Connect a vertex of a triangle to the 2nd Torricelli apex
	(In the literature, these things are often named after Pierre de Fermat.)
Touch	A line and a circle or two circles intersecting each other at exactly one point; that is, they do not cut through each other
	Touching Point — the point where a line and a circle or two circles touch
Transformation	A relation between two sets of points that is one-to-one and onto; that is, every point in one set is associated with exactly one point in the other set.
Transitive Relation	If a relation is true for a and b and for b and c, then it is true for a and c
Transversal	A line that is not parallel to either of two given lines
Traverse	Lateral rotation of a machine gun on its tripod
	Top — The maximum angle that a gun can traverse

Triangle	Segments connecting three noncollinear points, called vertices; *e.g.,* $\overline{EFG}$.	
	Acute	A triangle with all angles acute
	Antipedal	The triangle that a given triangle is pedal to relative to a given pedal point
	Contact	The pedal triangle if the incenter is the pedal point
	Crossed	Two transversals that intersect between two parallel lines; cut off the tails to see the triangles
	Degenerate	The vertices are collinear; this is not a triangle
	Double–Long	The triangle whose vertices are excenters; excentral
	Egyptian	A triangle with sides 3, 4 and 5 units long
	Equilateral	A triangle with all sides equal
	Half Equilateral	An equilateral triangle cut at its center line
	Isosceles	A triangle with two sides equal
	Long	The triangle whose vertices are long centers
	Medial	The three mid-segments of a triangle as its sides
	Nested	Two transversals that intersect outside two parallel lines; cut off the tails to see the triangles
	Obtuse	A triangle with one angle obtuse
	Orthic	Connect the feet of a triangle's altitudes
	Parent	The triangle from which a medial triangle is derived
	Pedal	The connected feet of perpendiculars dropped from the pedal point
	Right	A triangle with one angle right
	Scalene	A triangle with all unequal sides
	Tangential	Sides tangent to the circumcircle at the vertices of $\overline{EFG}$
	Too Obtuse	A triangle with one angle equal to or greater than 2φ
	Viviani	$\overline{EPP_F}$ and $\overline{FPP_E}$ given $\overline{EFG}$ isosceles with base $\overline{EF}$ and pedal point P anywhere between E and F
Trichotomy	Proof by contradiction when there are three alternatives	
Tri–Segment	Triangle	A segment connecting the trisection points, either both close to the base or both close to the apex
Undefined Terms	Intuitive concepts: plane, point, shortest path, straight	
Under Defined	Not enough given information; the solutions are infinite in number	
Vertex	The intersection of two lines, rays, or sides of a triangle or quadrilateral	
Visible Under an Angle	All of the points interior to an angle; field of fire	
Wallace Line	The pedal triangle is a line if the pedal point is on the circumcircle (In the literature, this is often called the Simson line.)	
Width	Given two parallel lines, the length of a perpendicular between them	
	Shoulder	The altitude of a rectangle's definitional triangle

Glosario inglés–español

Adjacent (Adyacente)	Dos triángulos disjuntos con un lado común (común para su longitud completa), o dos ángulos con una semirrecta común (vértice y dirección comunes)
Altitude (Altura)	La perpendicular desde un vértice de un triángulo hasta la extensión del lado opuesto
Analytic (Analítico)	Conocimiento contenido en la información dada
Angle (Ángulo)	Dos semirrectas, denominadas lados, que comparten un punto final común, denominado vértice. $\angle F$ si hay un ángulo en F o $\angle EFG$ para el ángulo entre $\overrightarrow{FE}$ y $\overrightarrow{FG}$

Acute (Agudo)	Un ángulo que es menor que un ángulo recto
Alternate Interior (Alterno interno)	Ángulos en lados opuestos de una transversal y entre las dos rectas dadas
Apex (Ápice)	El ángulo opuesto a la base de un triángulo
Base (Base)	En un triángulo con una base, los ángulos en cada extremo
Central (Central)	Un ángulo cuyo vértice es el centro de una circunferencia
Complementary (Complementario)	Dos ángulos que sumados forman un ángulo recto
Conjugate (Conjugado)	Ángulos que sumados forman dos ángulos llanos
Consecutive (Consecutivo)	Ángulos interiores (exteriores) en los extremos de un lado
Elevation (Elevación)	Una semirrecta está en una superficie plana y la otra está por encima de ésta
Exterior (Exterior)	El ángulo suplementario a un ángulo interior
Inscribed (Inscrito)	Un ángulo dentro de una circunferencia con su vértice en la circunferencia
Interior (Interior)	Un ángulo dentro de un triángulo o cuadrilátero en un vértice
Obtuse (Obtuso)	Un ángulo mayor que un ángulo recto y menor que un ángulo llano
Parallelism (Paralelismo)	En geometría hiperbólica; $2\,\mathrm{atan}(e^{-x})$ con altura x
Right (Recto)	La bisección de un ángulo llano
Skew (Sesgo)	La diferencia de los ángulos base de un triángulo
Straight (Llano)	Un ángulo cuyas semirrectas son colineales y opuestas
Supplementary (Suplementario)	Dos ángulos que sumados forman un ángulo llano
Vertical (Opuestos por el vértice)	Ángulos opuestos el uno al otro en una intersección

Anticenter (Anticentro)	El punto de concurrencia de las m-alturas; existe para cuadriláteros cíclicos

Apex (Ápice) El vértice opuesto a la base de un triángulo

Arc (Arco) Parte de una circunferencia; dentro de circunferencias iguales, los ángulos en el centro y los arcos que cortan son transformaciones los unos de los otros.

Area (Área) La medida del tamaño de un triángulo o de una unión de triángulos disjuntos

Auxiliary (Auxiliar) Rectas o arcos no dados cuya intersección va más allá de lo analítico

Axiom (Axioma) Una proposición que se asume sin pruebas para estudiar las consecuencias que se derivan de ella

Base (Base) El lado de un triángulo isósceles entre los ángulos iguales
El lado de un triángulo designado como tal, o sobre el que está construido

Between (Entre)
1. Si F está entre E y G, entonces F también está entre G y E y existe una recta que contiene los puntos E, F, G. (Entre implica que los tres puntos son distintos.)
2. Si E y G son dos puntos de una recta, entonces existe al menos un punto F entre E y G, y al menos un punto H tal que G queda entre E y H.
3. De cualesquiera tres puntos colineales, hay exactamente uno entre los otros dos.

Bi–Conditional (Bicondicional) Un enunciado de la forma p si y solo si q. Es verdadero si ambas p y q son verdaderas o ambas p y q son falsas. p implica q; también, q implica p. La prueba de ninguna de las implicaciones puede citar la otra implicación. Si y solo si se abrevia sii.

Bi–Medial (Bimedial) La intersección de las diagonales de un cuadrilátero

Bimedian (Bimediana) Un segmento que une los puntos medios de los lados opuestos de un cuadrilátero

Bisect (Bisectriz) $\frac{1}{2}$ Divide un segmento o un ángulo en dos partes iguales, denominadas mitades

Center Line (Línea central) La mediatriz de la base de un triángulo isósceles o de un semicírculo

Centroid (Centroide) El punto de equilibrio de una placa uniforme. Probar que el punto medial de un triángulo es su centroide requiere cálculo, por lo que estos términos *no* son intercambiables.

Chord (Cuerda) El segmento entre dos puntos de una circunferencia
Common (Común) El segmento entre los puntos de intersección de dos circunferencias

Circle (Circunferencia) Todos los puntos equidistantes de un punto, que se denomina centro

Circum	circle (Circunferencia circunscrita)	Una circunferencia que interseca una figura en sus vértices
	center (Circuncentro)	El centro de una circunferencia circunscrita
	radius (Circunradio)	El radio de una circunferencia circunscrita; R
Closing the Horizon (Cierre del horizonte)	La suma de todos los ángulos alrededor de un punto debe ser 2σ.	
Collinear (Colineal)	Un conjunto de puntos que están todos en la misma recta	
Concentric (Concéntrico)	Dos o más circunferencias con el mismo centro, pero con diferentes radios	
Concurrent (Concurrente)	Tres o más rectas o arcos que se cruzan en el mismo punto	
Concyclic (Cocíclico)	Cuatro o más puntos en la misma circunferencia	
Condition (Condición)	Restricciones que una figura cumple o no	
Conformal (Conforme)	Una transformación que conserva los ángulos; p. ej., escalado e inversión de la circunferencia	
Congruent (Congruente) $\cong$	Dos triángulos cuyas áreas, lados y ángulos interiores son iguales	
Contradiction, Proof by (Contradicción, Prueba por)	Para probar que el enunciado p implica el enunciado q, se asume que p es verdadero y que q no es verdadero, y se demuestra que esto es imposible.	
Converse (Opuesto)	Dado el argumento que p implica q, el argumento que q implica p	
Convex (Convexo)	Cualquier segmento entre dos puntos interiores a dos lados está dentro de la figura	
Defect (Defecto)	En geometría hiperbólica; $\sigma - (\alpha + \beta + \gamma)$ para un triángulo de un tamaño dado	
Diagonal (Diagonal)	Segmentos que conectan vértices no consecutivos de cuadriláteros	
	Definitional (Definitiva)	El lado adyacente de los dos triángulos de un cuadrilátero
Diameter (Diámetro)	Una cuerda que cruza el centro de una circunferencia	
	Diametrically Opposed (Diametralmente opuesto)	Los extremos de un diámetro

Dichotomy (Dicotomía)		Prueba por contradicción cuando hay dos alternativas
Discussion (Discusión)		Las condiciones necesarias y suficientes para una solución, y cuántas soluciones
Disjoint (Disjunto)		Figuras que no se superponen; sus áreas forman un grupo aditivo (Incluye circunferencias que se tocan y triángulos adyacentes, si están fuera el uno del otro).
Disjoint (Disjunto)		La probabilidad de que algún punto esté dentro de ambas figuras es cero
Edubabble (Balbuceos Educativos)		Palabras ridículamente esponjosas y consignas tontas destinadas a ofuscar y confundir (los profesores estadounidenses de secundaria no estudian su materia en la universidad – ellos seleccionan de los libros de texto de sus estudiantes – en su lugar, reciben balbuceos educativos).
Endpoint (Extremo)		Un punto al final de un segmento, arco o semirrecta
Equal (Igual) $=$		Magnitudes comparables que no son ni menores ni mayores la una que la otra
Equidistant (Equidistante)		Dos pares de puntos que definen dos segmentos de igual longitud
	Lines (Rectas)	Cualesquiera dos perpendiculares entre ellas son de igual longitud
Equivalence (Equivalencia)	**Class (Clase de)**	Un conjunto de objetos que son iguales, congruentes, semejantes o paralelos
	Relation (Relación de)	Un conjunto y una relación reflexiva, simétrica y transitiva
Equivalent (Equivalente)		Cualesquiera dos condiciones que son bicondicionales
Euler (Euler)	**Center (Centro)**	El centro de la circunferencia de Euler
	Circle (Circunferencia de)	De un triángulo, la circunferencia a través de los puntos medios de los lados, los pies de las alturas y los puntos medios desde el ortocentro a los vértices. En algunos libros de texto, es llamada circunferencia de los nueve puntos.
	Segment (Segmento de)	El segmento desde el ortocentro hasta el circuncentro
Ex	**circle (Circunferencia exinscrita)**	Una circunferencia tangente a un lado de un triángulo y a las extensiones de los lados adyacentes
	center (Exicentro)	Un centro de una circunferencia exinscrita
	foot (Pie bisector exterior)	Donde una bisectriz de un ángulo exterior corta la extensión del lado opuesto; $E^{\times}, F^{\times}, G^{\times}$

	radii (Exradios)	Los radios de las circunferencias exinscritas; r_X, r_Y, r_Z son de $\omega_X, \omega_Y, \omega_Z$
Extend (Extiende)	Dado $\overline{EF}$, construye $\overline{EG}$ de modo que F esté dentro de $\overline{EG}$ o E esté dentro de $\overline{FG}$.	
Field of Fire (Campo de tiro)	Todos los puntos interiores al ángulo de tiro horizontal de una ametralladora	
Figure (Figura)	Un conjunto de puntos. Pueden estar solos o unidos en rectas, segmentos o arcos.	
Foot (Pie)	La intersección cuando se traza una perpendicular desde un punto a una recta	
Frustum, Triangle (Tronco del triángulo)	La parte de un triángulo entre la base y un corte paralelo a la base	
Fully Defined (Completamente definido)	Una figura con las características dadas existe y es única	
Geometric Mean (Media geométrica)	Si $\frac{a}{b} = \frac{b}{c}$ para números reales a, b, c, entonces b es la media geométrica de a y c	
Half–Scale (A media escala)	Un triángulo cuyos lados son la mitad de los lados correspondientes de otro triángulo	
Harmonic Division (División armónica)	Corta un segmento $\overline{EF}$ interna y externamente en la misma proporción; $\frac{\overline{EG^*}}{\overline{FG^*}} = \frac{\overline{EG^\times}}{\overline{FG^\times}}$	
Homothetic	Center (Centro de homotecia)	$\overleftrightarrow{E''E} \cap \overleftrightarrow{F''F} \cap \overleftrightarrow{G''G}$ para $\overline{E''F''G''}$ doble homotético de $\overline{EFG}$
	Double (Doble homotético)	Un triángulo cuyos lados miden el doble de la longitud de los lados de otro triángulo y cuyas extensiones laterales son paralelas por pares a las extensiones laterales de ese triángulo
	Triple (Triple homotético)	Análogo a un doble homotético, pero de longitud triple
Hypotenuse (Hipotenusa)	El lado opuesto al ángulo recto de un triángulo rectángulo	
In	circle (Circunferencia inscrita)	Una circunferencia que toca cada lado de una figura
	center (Incentro)	El centro de la circunferencia inscrita
	diameter (Diámetro)	El diámetro de la circunferencia inscrita; d
	foot (Pie bisector interior)	Donde la bisectriz de un ángulo corta el lado opuesto; E^*, F^*, G^*

	Radius (Inradio)	El radio de una circunferencia inscrita; r
Inside (Interior a)	Segment (Segmento)	Un miembro del conjunto de puntos de un segmento, pero no un extremo
	Figure (Figura)	Un punto tal que cualquier recta que lo atraviese corta la figura exactamente en dos puntos, y el punto está entre ellos
	Triangles (Triángulos)	Un triángulo cuyos puntos están dentro o sobre un lado de otro triangulo, pero los triángulos no coinciden
Isometric (Isométrica)	Una transformación que conserva las longitudes; por LLL, también conserva los ángulos	
Kill (Muerte)	Chord (Cuerda de la muerte)	Un segmento desde un lado de un ángulo hasta el otro
	Circle (Círculo de la muerte)	La circunferencia de mayor radio que toca o está en el interior de dos ángulos
Legs (Patas o catetos)	Triangle (de un triángulo)	Los lados que no son la base o la hipotenusa
	Triangle Frustum (del tronco de un triángulo)	Los lados que no son paralelos
Lemma (Lema)	Un teorema usado para probar otros teoremas más importantes	
Length (Longitud)	La medida del tamaño de un segmento; la distancia entre sus extremos	
Line (Recta)	Un segmento extendido más allá de ambos extremos; denotado $\overleftrightarrow{EF}$ si $\overline{EF}$ es el segmento	
Line of Centers (Recta de centros)	La recta que pasa por los centros de dos circunferencias	
Lines, Supplementary (Rectas suplementarias)	Si los ángulos alternos internos son suplementarios, las dos rectas cruzadas por la transversal son suplementarias con relación a esa transversal.	
Locus (Lugar geométrico)	Todos los puntos que satisfacen una condición; el plural es loci, en inglés	
Long (Distante)	Dentro de un ángulo interior de un triángulo, pero fuera del triángulo Dentro del campo de tiro de una ametralladora, pero más allá de su cuerda de la muerte	
	Center (Centro)	La intersección de una bisectriz de un ángulo y la circunferencia circunscrita

	Circle (Circunferencia)	Alrededor de un centro distante a través del incentro y el exicentro
Magnitude (Magnitud)		Un conjunto con una relación de equivalencia entre sus miembros, $=$, y un orden total, $\leq$
Maltitude (M-altura)		Punto medio-altura; la perpendicular trazada desde el punto medio de un lado de un cuadrilátero hasta el lado opuesto
Measure (Medida)		El tamaño de los conjuntos; número de puntos discretos, longitud de segmentos, o áreas de triángulos o de la unión de triángulos disjuntos.
Medial Point (Punto medial)		El punto donde las medianas o las bimedianas concurren
Median (Mediana)		Un segmento desde un vértice de un triángulo hasta el punto medio del lado opuesto
Mediator (Mediatriz)		La bisectriz perpendicular de un segmento
Midpoint (Punto medio)		El punto donde se divide un segmento en dos partes iguales
Mirror Property (Propiedad del espejo)		La luz que rebota de un espejo a un punto va tan lejos como lo haría a su reflejo
Mid–Segment (Segmento medio)	Triangle (de un triángulo)	Un segmento que conecta los puntos medios de dos lados
	Triangle Frustum (del tronco de un triángulo)	Un segmento que conecta los puntos medios de las patas
Miquel (Miquel)	Point (Punto de)	El punto definido por el teorema de Miquel
	Circles (Circunferencias de)	Las circunferencias circunscritas definidas por el teorema de Miquel
Napoleon Point (Punto de Napoleón)	First (Primer punto de Napoleón)	El centro del triángulo equilátero definido en el teorema de Napoleón para triángulos equiláteros externos
	Second (Segundo punto de Napoleón)	Igual, pero para los triángulos equiláteros internos
Neutral Geometry (Geometría Neutra)		Un conjunto de postulados que no hace referencia a rectas paralelas; geometría absoluta
Non–Euclidean (No euclidiano)		Un conjunto de postulados que contiene uno que contradice el postulado de las paralelas
Opposite (Opuesto)	In a Triangle (En un tríangulo)	Un ángulo y un lado opuestos el uno al otro

	Of a Line (De una recta)	Extremos de un segmento cruzado por la recta
	In a Quadrilateral (En un cuadrilátero)	Dos lados o dos ángulos opuestos
Ordering (Orden)	Un conjunto y una relación, $\leq$, que es reflexiva, antisimétrica y transitiva	
	Total (Total)	$a \leq b$ o $b \leq a$ para cada a, b en el conjunto
Orthic Reflection (Reflexión del ortocentro)	La reflexión del ortocentro alrededor de un lado del triángulo	
Orthocenter (Ortocentro)	El punto donde las alturas de un triángulo son concurrentes	
Orthogonal (Ortogonal)	Dos arcos cuyas tangentes en su intersección son perpendiculares	
Parallel (Paralelo)	Dos rectas que no se cruzan	
Pedal Point (Punto podal)	Un punto desde el cual se trazan perpendiculares a los lados o las extensiones de los lados de un triángulo o un cuadrilátero	
Pencil (Lápiz)	Un triángulo con los pies bisector interior y bisector exterior de la base conectados al ápice. (Lápiz es una gran palabra en geometría avanzada, pero todavía no hemos llegado allí. Solamente dibujar esta figura y llamarla lápiz es suficiente para un estudiante de secundaria).	
Perimeter (Perímetro)	La suma de las longitudes de los lados de un triángulo o cuadrilátero	
Perpendicular (Perpendicular)	Una recta cuya intersección con otra recta forma un ángulo recto	
Polygon (Polígono)	La unión de varios triángulos adyacentes en sus lados de modo que sea convexa	
Postulate (Postulado)	Los axiomas que son específicos de la geometría, no de otras ramas de la matemática	
Power of the Point (Potencia de un punto)	Para un punto P y una circunferencia con centro O y radio r; si $z = \lvert\overline{OP}\rvert$, entonces la potencia del punto es $\lvert P\rvert = \lvert r^2 - z^2\rvert$. Si las distancias de P a la circunferencia a través de una cuerda o una secante son x e y, entonces $xy = \lvert P\rvert$.	
Probability (Probabilidad)	La razón de la medida de un subconjunto a la medida de todo el conjunto	
Projection (Proyección)	En $\overline{EFG}$, la proyección de $\overline{EG}$ y $\overline{FG}$ hacia $\overline{EF}$ es $\overline{EG'}$ y $\overline{FG'}$, respectivamente	

Quadrature (Cuadratura)	Teoremas que demuestran la igualdad de las áreas de triángulos o de uniones de triángulos
Quadrilateral (Cuadrilátero)	La unión de dos triángulos adyacentes en un lado de modo que sea convexa; $\overline{EFGH}$

Bi–Centric (Bicéntrico)	Un cuadrilátero que es tanto cíclico como tangencial
Contact (de contacto)	Los puntos de contacto de la circunferencia inscrita de un cuadrilátero tangencial conectados entre sí consecutivamente
Cyclic (Cíclico)	Un cuadrilátero para el cual existe una circunferencia circunscrita
Isosceles Kite (Deltoide isósceles)	Un deltoide con un par de lados iguales a la diagonal
Kite (Deltoide)	La unión de dos triángulos congruentes; los lados no comunes que son iguales también son consecutivos.
Lambert (de Lambert)	Un cuadrilátero con tres ángulos rectos
Long (Distante)	El cuadrilátero cuyos vértices son centros distantes
Medial Parallelogram (Paralelogramo medial)	Los puntos medios conectados de los lados consecutivos de un cuadrilátero
Orthodiagonal (Ortodiagonal)	Un cuadrilátero cuyas diagonales son perpendiculares
Parallelogram (Paralelogramo)	La unión de dos triángulos congruentes; los lados no comunes que son iguales también son opuestos.
Parent (Padre)	El cuadrilátero alrededor de un paralelogramo medial
Pedal (Podal)	La conexión de los pies de las perpendiculares trazadas desde el punto podal
Rectangle (Rectángulo)	Un cuadrilátero con ángulos iguales
Rhombus (Rombo)	Un cuadrilátero con todos los lados iguales; plural, rhombi, en ingles
Right Kite (Deltoide recto)	Un deltoide cuyos dos triángulos congruentes son rectángulos
Right Rectangle (Rectángulo recto)	Un rectángulo con ángulos rectos
Right Square (Cuadrado recto)	Un rectángulo recto con lados iguales
Saccheri (de Saccheri)	Un cuadrilátero con dos lados opuestos iguales y perpendiculares a la base
Square (Cuadrado)	Un rectángulo con lados iguales
Tangential (Tangencial)	Un cuadrilátero para el cual existe una circunferencia inscrita

Radius (Radio)	Un segmento desde el centro de la circunferencia hasta la circunferencia; plural, radii, en inglés
Random (Aleatorio)	Los puntos en un segmento o dentro de un triángulo o circunferencia están distribuidos uniformemente

Ray (Semirrecta)	Un segmento extendido en una dirección; denotado $\overrightarrow{EF}$ si $\overline{EF}$ es el segmento	
Reflection (Reflexión)	Desde un punto, dibuja una recta a través de un punto y la extiende hasta una distancia igual Desde un punto, lo refleja a través del pie de una perpendicular trazada en una recta Desde una recta, refleja dos puntos a través de un punto y dibuja una línea a través de ellos Desde una circunferencia, refleja su centro a través de un punto y dibuja una circunferencia idéntica	
Reflexive Relation (Relación reflexiva)	Una relación binaria sobre un conjunto tal que cada elemento está relacionado consigo mismo	
Relation (Relación)	Un operador de verdadero/falso en un par ordenado de elementos de un conjunto dado	
Secant (Secante)	Una recta que corta una circunferencia en exactamente dos puntos	
Segment (Segmento)	Todos los puntos a lo largo del camino más corto entre dos puntos; $\overline{EF}$	
Semi (Semi)	Difference (Semidiferencia)	La mitad de la diferencia entre dos longitudes o entre dos ángulos
	Perimeter (Semiperímetro)	La mitad del perímetro de un triángulo
	Sum (Semisuma)	La mitad de la suma de dos longitudes o de dos ángulos
Side (Lado)	Triangle (de un triángulo)	Uno de los tres segmentos que forman un triángulo
	Quadrilateral (de un cuadrilátero)	Un segmento no común de uno de sus triángulos
	Consecutive (Consecutivo)	Los lados de un cuadrilátero que comparten un extremo
Similar (Semejante) $\sim$	Dos triángulos con todos sus ángulos correspondientes iguales	
Steiner Line (Recta de Steiner)	Una recta paralela a la recta de Wallace tal que la recta de Wallace está en la mitad entre aquella y el punto podal de la recta de Wallace; también llamada recta ortocéntrica.	
Subtend (Subtender)	Crear una cuerda una clase de equivalencia de todos los ángulos inscritos a un lado de esta	
Subtend at Center (Subtender en el centro)	Formar una cuerda un ángulo con su vértice en el centro de la circunferencia	

Summit (Cumbre) El lado opuesto a la base de un cuadrilátero de Saccheri

Symmetric Relation (Relación simétrica) Una relación que puede establecerse entre dos cosas en cualquier orden

Synthetic (Sintético) Conocimiento que permanece después de que se borran las rectas y los arcos auxiliares

T & V (T y O) Los teoremas de las transversales y los ángulos opuestos por el vértice, utilizados en alguna combinación

Tangent (Tangente) Una recta que toca una circunferencia; si se hace referencia a su longitud, esta significa la longitud del segmento entre el punto de contacto y el punto que define la recta tangente.

- **Cut (Corte)** El segmento de una tangente interna que está entre las tangentes externas
- **External (Externa)** Una recta tangente a dos circunferencias que no pasa entre sus centros, o el segmento entre los puntos de contacto
- **Internal (Interna)** Una recta tangente a dos circunferencias disjuntas que pasa entre sus centros, o el segmento entre los puntos de contacto

Theorem (Teorema) Un enunciado que requiere pruebas utilizando postulados o teoremas ya probados

Torricelli (Torricelli) Estos términos se aplican solo a triángulos con ángulos menores que 2φ.

- **Apex (Ápice de)** El ápice de un triángulo equilátero construido sobre el exterior de un lado de un triángulo; dado $\overline{EFG}$, E'' es opuesto a E.
- **2nd Apex (2.º ápice de)** El ápice de un triángulo equilátero construido sobre el interior de un lado de un triángulo; en este libro, no está etiquetado.
- **Point (Punto de)** El punto de concurrencia de los segmentos de Torricelli; U
- **2nd Point (2.º punto de)** El punto de concurrencia de los 2.os segmentos de Torricelli; V
- **Segment (Segmento de)** Conecta un vértice de un triángulo con el ápice de Torricelli opuesto a este; p. ej. $\overline{EE''}, \overline{FF''}, \overline{GG''}$
- **2nd Segment (2.º segment de)** Conecta un vértice de un triángulo con el 2.º ápice de Torricelli

(En las publicaciones, a estos elementos a menudo se les da el nombre de Pierre de Fermat).

Touch (Tocar) Intersecarse una recta y una circunferencia o dos circunferencias exactamente en un punto, es decir, sin cruzarse

	Touching Point (Punto de contacto)	El punto en el que una recta y una circunferencia o dos circunferencias se tocan
Transformation (Transformación)	Una relación entre dos conjuntos de puntos que es uno a uno y sobreyectiva; es decir, cada punto de un conjunto está asociado con exactamente un punto del otro conjunto.	
Transitive Relation (Relación transitiva)	Si una relación es verdadera para a y b y para b y c, entonces es verdadera para a y c	
Transversal (Transversal)	Una recta que no es paralela a ninguna de las dos rectas dadas	
Traverse (Rotación horizontal)	Rotación lateral de una ametralladora sobre su trípode	
	Top (Ángulo de tiro horizontal)	El ángulo máximo en que puede rotar una ametralladora
Triangle (Triángulo)	Segmentos que conectan tres puntos no colineales, denominados vértices; p. ej. $\overline{EFG}$.	
	Acute (Agudo)	Un triángulo con todos los ángulos agudos
	Antipedal (Antipodal)	El triángulo con el que un triángulo dado es podal en relación con un punto podal dado
	Contact (de contacto)	El triángulo podal si el incentro es el punto podal
	Crossed (Cruzado)	Dos transversales que se cruzan entre dos rectas paralelas; cortar las colas para ver los triángulos
	Degenerate (Degenerado)	Los vértices son colineales; esto no es un triangulo
	Double–Long (Doble distante)	El triángulo cuyos vértices son exicentros; excéntrico
	Egyptian (Egipcio)	Un triángulo con lados de 3, 4 y 5 unidades de largo
	Equilateral (Equilátero)	Un triángulo con todos los lados iguales
	Half Equilateral (Medio equilátero)	Un triángulo equilátero cortado por su línea central
	Isosceles (Isósceles)	Un triángulo con dos lados iguales
	Long (Distante)	El triángulo cuyos vértices son centros distantes
	Medial (Medial)	Los tres segmentos medios de un triángulo como sus lados
	Nested (Anidado)	Dos transversales que se intersecan fuera de dos rectas paralelas; cortar las colas para ver los triángulos
	Obtuse (Obtuso)	Un triángulo con un ángulo obtuso
	Orthic (Órtico)	Conecta los pies de las alturas de un triángulo
	Parent (Padre)	El triángulo desde el cual se deriva un triángulo medial
	Pedal (Podal)	La conexión de los pies de las perpendiculares trazadas desde el punto podal
	Right (Rectángulo)	Un triángulo con un ángulo recto

	Scalene (Escaleno)	Un triángulo con todos los lados desiguales
	Tangential (Tangencial)	Lados tangentes a la circunferencia circunscrita en los vértices de $\overline{EFG}$
	Too Obtuse (Demasiado obtuso)	Un triángulo con un ángulo igual o mayor que 2φ
	Viviani (Viviani)	$\overline{EPP_F}$ y $\overline{FPP_E}$ dado $\overline{EFG}$ isósceles con base $\overline{EF}$ y punto podal P en cualquier lugar entre E y F
Trichotomy (Tricotomía)	Prueba por contradicción cuando hay tres alternativas	
Tri–Segment (Trisegmento)	**Triangle (de un triángulo)**	Un segmento que conecta los puntos de una trisección, ambos cerca de la base o ambos cerca del ápice
Undefined Terms (Términos no definidos)	Conceptos intuitivos: plano, punto, camino más corto, llano	
Under Defined (Mal definidos)	No hay suficiente información dada; las soluciones son infinitas en número	
Vertex (Vértice)	La intersección de dos rectas, semirrectas o lados de un triángulo o cuadrilátero	
Visible Under an Angle (Visible bajo un ángulo)	Todos los puntos interiores a un ángulo; campo de tiro	
Wallace Line (Recta de Wallace)	El triángulo podal es una recta si el punto podal está sobre la circunferencia circunscrita (En las publicaciones, esto a menudo se denomina la recta de Simson)	
Width (Ancho)	Dadas dos rectas paralelas, la longitud de una perpendicular entre ellas	
	Shoulder (de hombro)	La altura del triángulo de definición de un rectángulo

Das englisch–deutsche Glossar

Adjacent (Adjazent) Zwei disjunkte Dreiecke mit einer gemeinsamen Seite (gemeinsam für volle Länge) oder zwei Winkel mit einem gemeinsamen Strahl (gemeinsame Scheitel und Richtung)

Altitude (Höhe) Die Senkrechte von einer Dreiecksecke auf die Verlängerung der gegenüberliegenden Dreiecksseite

Analytic (Analytisch) In der gegebenen Informationen enthaltenes Wissen

Angle (Winkel) Zwei Strahlen, die Schenkel des Winkels heißen, haben einen gemeinsamen Anfangspunkt und bilden einen Scheitel. $\angle F$, wenn F ein Winkel ist, oder $\angle EFG$ für einen Winkel zwischen $\overrightarrow{FE}$, $\overrightarrow{FG}$

Acute (Spitzer)	Ein Winkel kleiner als ein rechter Winkel
Alternate Interior (Wechselwinkel)	Ein Winkel auf den unterschiedlichen Seiten von einer Geraden, die zwei Parallelen schneidet
Apex (Spitze)	Ein Winkel, der der Basis eines Dreiecks gegenüber steht
Base (Basis)	Beide Winkel, die an der Basis eines Dreiecks anliegen
Central (Mittelpunktswinkel)	Ein Winkel, dessen Spitze im Mittelpunkt des Kreises sich befindet
Complementary (Komplementwinkel oder Komplementärwinkel)	Zwei Winkel, die sich zu einem rechten Winkel ergänzen
Conjugate (Ergänzungswinkel)	Winkel, die sich zu zwei gestreckten Winkel ergänzen
Consecutive (Nachbarwinkel)	Innen- oder Außenwinkel, die auf derselben Seite liegen
Elevation (Höhenwinkel)	Ein Winkel zwischen einer Strahl auf der Bodenebene und der anderen darüberligenden Strahl.
Exterior (Außenwinkel)	Ein Nebenwinkel eines Innenwinkels
Inscribed (Kreiswinkel)	Ein Winkel innerhalb eines Kreises mit seinem Eckpunkt auf dem Kreis
Interior (Innenwinkel)	Ein Winkel innerhalb eines Dreiecks oder Vierecks an einem Scheitel
Obtuse (Stumpfer)	Ein Winkel größer als ein rechter Winkel und kleiner als ein gestreckter Winkel
Parallelism (Parallelität)	In hyperbolischer Geometrie; $2\,\mathrm{atan}(e^{-x})$, wo x eine Höhe ist
Right (Rechter)	Ein Winkel, der durch Halbierung des gestreckten Winkels entsteht

Skew (Schräg)	Eine Differenz zwischen Basiswinkeln eines Dreiecks
Straight (Gestreckter)	Ein Winkel, dessen Schenkel kollinear sind und eine Gerade bilden
Supplementary (Supplement)	Zwei Winkel, die sich zu einem gestreckten Winkel ergänzen
Vertical (Scheitelwinkel oder Gegenwinkel)	Die sich gegenüber liegenden Winkel, die entstehen, wenn sich zwei Geraden schneiden

Anticenter (Antizentrum)	Der Punkt, an dem sich die Mittellinien treffen; es gibt für zyklische Vierecke
Apex (Spitze)	Der Basis gegenüberliegende Eckpunkt des Dreiecks
Arc (Kreisbogen)	Teil eines Kreises; innerhalb gleicher Kreise sind die Mittelpunktswinkel und die von ihnen abgeschnittenen Bögen eine Transformation voneinander.
Area (Fläche)	Das Maß für die Größe eines Dreiecks oder einer Vereinigung der disjunkten Dreiecke
Auxiliary (Zusätzlich)	Nicht vorgegebene Linien oder Bögen, deren Schnittpunkt über die Grenzen der Analytik hinausgeht
Axiom (Axiom)	Ein Satz, der beweislos vorausgesetzt wird um die Folgen daraus abzuleiten
Base (Basis oder Grundseite)	Die Seite eines gleichschenkligen Dreiecks, die zwischen den gleichgroßen Basiswinkeln liegt Die Seite eines Dreiecks, die als solche bezeichnete oder die Seite, auf der das Dreieck aufgebaut wird.
Between (Zwischen)	1. Wenn F zwischen E und G liegt, liegt F auch zwischen G und E, und es existiert eine Gerade, die die Punkte E, F, G enthält. (Zwischen impliziert, dass die drei Punkte getrennt voneinander sind.) 2. Wenn E und G zwei Punkte auf einer Gerade sind, existiert es mindestens ein Punkt F, der zwischen E und G liegt, und mindestens ein Punkt H, der so angeordnet ist, dass G zwischen E und H liegt. 3. Von etwaigen drei kollinearen Punkten befindet sich genau einer zwischen den beiden anderen.
Bi–Conditional (Bikonditional)	Eine Aussage in der Form p dann und nur dann q. Sie ist genau dann wahr, wenn sowohl p als auch q wahr sind oder beide falsch sind. q folgt aus p; also, p folgt aus q. Im Beweis einer Implikation kann man nicht die andere verwenden und umgekehrt. Dann und nur dann wird als iff abgekürzt.
Bi–Medial (Diagonalenschnittpunk)	Der Schnittpunkt der Diagonalen eines Vierecks

Bimedian (Bimedian)	Eine Strecke, die die Mittelpunkte der gegenüberliegenden Seiten eines Vierecks verbindet
Bisect (Halbieren) $\frac{1}{2}$	Ein Abschnitt oder ein Winkel in zwei gleich große Teile wird zerlegt, die man Hälften nennt
Center Line (Schwerlinie)	Die Seitenhalbierende der Basis eines gleichschenkligen Dreiecks oder ein Halbkreis
Centroid (Zentroid oder Schwerpunkt)	Der Gleichgewichtspunkt einer einheitlichen Fläche. Wenn man behauptet, dass der Mittelpunkt eines Dreiecks sein Schwerpunkt ist, ist Kalkül erforderlich, d.h. diese Ausdrücke sind *nicht* austauschbar.
Chord (Sehne)	Die Strecke, die zwei Punkte eines Kreises verbindet **Common (Potenzlinie oder Chordale)** — Die Strecke zwischen den Schnittpunkten zweier Kreise
Circle (Kreis)	Die Menge aller Punkte, die einen konstanten Abstand zu einem Punkt haben, der Mittelpunkt heißt
Circum (Um)	**circle (Umkreis)** — Ein Kreis, der durch alle Eckpunkte einer Figur geht **center (Umkreismittelpunkt)** — Das Zentrum des Umkreises **radius (Halbmesser)** — Der Radius des Umkreises; R
Closing the Horizon (Schließung des Horizonts)	Die Summe aller Winkel muss um einen Punkt gleich 2σ sein.
Collinear (Kollinear)	Eine Menge von Punkten, die alle auf ein und derselben Geraden liegen
Concentric (Konzentrisch)	Zwei oder mehr Kreise, die sich ein und denselben Schwerpunkt besitzen, jedoch unterschiedliche Radien (Halbmesser) aufweisen
Concurrent (Kopunktal)	Drei oder mehr Geraden oder Bögen, die durch einen gemeinsamen Punkt gehen
Concyclic (Konzyklisch)	Vier oder mehr Punkte, die auf dem Rand eines Kreises liegen
Condition (Kondition)	Die Voraussetzungen, die eine Figur entweder erfüllt oder nicht
Conformal (Konform)	Eine winkeltreue Abbildung; z.B. Skalierung und Kreisspiegelung
Congruent (Kongruent) $\cong$	Zwei Dreiecke, deren Flächen, Seiten und Innenwinkel gleich sind
Contradiction, Proof by (Beweis durch Widerspruch)	Wir beweisen, dass aus p q folgt, indem wir annehmen, dass p wahr und q falsch ist, dass zu einem Widerspruch führt.

Converse (Gegenteilig)	Wenn p q impliziert, gibt es die Aussage, dass q p impliziert	
Convex (Konvex)	Ein beliebiger Abschnitt zwischen zwei Punkten, die sich innerhalb von zwei Seiten befinden, liegt innerhalb der Figur	
Defect (Defekt)	In hyperbolischer Geometrie; $\sigma - (\alpha + \beta + \gamma)$ für ein Dreieck einer bestimmten Größe	
Diagonal (Diagonale)	Die Verbindungslinien zwischen nicht benachbarten Ecken eines Vierecks	
	Definitional (Definierende)	Die benachbarte Seite der beiden Dreiecke in einem Viereck
Diameter (Durchmesser)	Eine Sehne, die durch den Mittelpunkt verläuft	
	Diametrically Opposed (Genau entgegengesetzt)	Die Endpunkte eines Durchmesser
Dichotomy (Dichotomie)	Beweis durch Widerspruch, wenn es zwei Ausweichmöglichkeiten gibt	
Discussion (Diskussion)	Die notwendigen und hinreichenden Bedingungen für eine Lösung, und die Anzahl solcher Lösungen	
Disjoint (Disjunkt)	Die Figuren, die sich nicht überlappen; Ihre Fläche bilden eine additive Gruppe. Dies beinhaltet die Berührungskreise und benachbarte Dreiecken.	
Disjoint (Disjunkt)	Es besteht keine Wahrscheinlichkeit, dass einige Punkte innerhalb beider Figuren liegen.	
Edubabble (Bildungs-Geschwurbel)	Lächerlich luftige Worte und alberne Abfassung von Lösungen, die dazu bestimmt sind, in die Irre zu führen und zu verwirren. (Amerikanische Oberlehrern studieren nicht ihr Fach in der Universität- Sie werden es vom Lehrbuch für ihre Schülern lernen - Anstatt erhalten sie sich fachliche Ausbildung.)	
Endpoint (Endpunkt)	Ein Punkt am Ende einer Gerade, eines Bogens oder Strahls	
Equal (Gleich) $=$	Vergleichbare Maße, die nicht kleiner oder größer als einander sind	
Equidistant (Längentreu oder Abstandstreu)	Zwei Paare von Punkten, die zwei gleich lange Segmente bestimmen	
	Lines (Geraden)	Jedes Paar der Senkrechten zwischen ihnen sind gleich lang
Equivalence (Equivalenz)	Class (Klasse)	Eine Menge von Objekten, die gleich, kongruent, ähnlich oder parallel sind
	Relation (Relation)	Eine Menge und eine Relation, die reflexiv, symmetrisch und transitiv ist
Equivalent (äquivalent)	Bedingungen, von denen zwei bikonditional sind	

Euler (Euler)	**Center (Mittelpunkt)**	Der Mittelpunkt eines Eulerkreises
	Circle (Eulerkreis)	Der Kreis, der die Mittelpunkte der Seiten, die Höhe fußen und die Mittelpunkte zwischen das Ortho-Zentrum und die Eckpunkte um einen Dreieck schneidet. Er ist auch genannt als neun-punkte-kreis in einigen Lehrbuchen.
	Segment (Euler-Gerade)	Eine Gerade, die den Höhenschnittpunkt mit dem Umkreismittelpunkt verbindet
Ex (Auß)	**circle (Ankreis)**	Der Kreis, der von einer Dreiecksseite von außen und von den Verlängerungen der beiden anderen Dreiecksseiten tangential berührt wird
	center (Ankreismittelpunkt)	Der Mittelpunkt des Ankreises
	foot (Äußere Winkelhalbierende)	Wenn die Winkelhalbierende die Verlängerung der gegenüberliegenden Seite schneidet; $E^\times, F^\times, G^\times$
	radii (Ankreishalbmesser)	Die Halbmesser der Ankreise; r_X, r_Y, r_Z jeweils von $\omega_X, \omega_Y, \omega_Z$

Extend (Verlängern)	Wenn $\overline{EF}$ gegeben ist, konstruieren Sie $\overline{EG}$ so, dass F innerhalb von $\overline{EG}$ liegt oder E innerhalb von $\overline{FG}$ liegt
Field of Fire (Schießfeld)	Alle Punkte, die innerhalb der maximalen Querdrehung des Maschinengewehrs liegen Geometer sagen “sichtbar unter einem Winkel”, ohne dass sie auf Waffen verweisen.
Figure (Figur)	Eine Menge von Punkten. Sie können alleine sein oder in Linien, Abschnitten oder Bögen gruppiert werden.
Foot (Fußpunkt)	Der Schnittpunkt einer Senkrechten von einem Punkt mit einer Geraden
Frustum, Triangle (abgeschnittenes Dreieck)	Der Teil eines Dreiecks zwischen der Basis und dem Schnitt parallel zur Basis
Fully Defined (Voll definiert)	Eine Figur mit den gegebenen Eigenschaften existiert und sie ist einzigartig.
Geometric Mean (Geometrisches Mittel)	Wenn $\frac{a}{b} = \frac{b}{c}$ für reelle Zahlen a, b, c gilt, dann ist b das geometrische Mittel von a und c.

Half–Scale (Halbe Skalierung)	Ein Dreieck, dessen Seiten halber Länge der entsprechenden Seiten eines anderen Dreieck entsprechen	
Harmonic Division (Harmonische Teilung)	Teilt eine Strecke $\overline{EF}$ auf zwei Teile, sodass das Teilverhältnis vom kleineren Teil zum größeren Teil den gleichen Betrag hat, wie das größere Teil zur ganzen Strecke $\overline{EF}$; $\frac{\overline{EG^*}}{\overline{FG^*}} = \frac{\overline{EG^\times}}{\overline{FG^\times}}$	
Homothetic (Homothetisch)	**Center (Ähnlichkeitspunkt)**	$\overleftrightarrow{E''E} \cap \overleftrightarrow{F''F} \cap \overleftrightarrow{G''G}$ für $\overline{E''F''G''}$ ist ein homothetisches Doppel $\overline{EFG}$
	Double (Doppel)	Ein Dreieck, dessen Seitenlängen zweimal langer als die Seitenlänge anderes Dreiecks sind, und dessen seitliche Erweiterungen paarweise parallel zu seitlichen Erweiterungen entsprechendes Dreiecks sind.
	Triple (Tripel)	Analog zu einem homothetischen Doppel, aber dreifach so lang
Hypotenuse (Hypotenuse)	Eine Seite eines rechtwinkligen Dreiecks, die dem Rechtwinkel gegenüber liegt	
In (Inn)	**circle (Inkreis)**	Ein Kreis, der jede Seite einer Figur in ihrem Inneren berührt
	center (Inkreismittelpunkt)	Das Zentrum des Inkreises
	diameter (Inkreisdurchmesser)	Der Durchmesser des Inkreises; d
	foot (Innere Winkelhalbierende)	Wenn die Winkelhalbierende die gegenüberliegende Seite schneidet; E^*, F^*, G^*
	radius (Inkreishalbmesser)	Der Halbmesser des Inkreises; r
Inside (Innerhalb)	**Segment (Segment)**	Eine Komponente der Menge von Segmentpunkten außer den Endpunkten
	Figure (Figur)	Ein Punkt so dass, jede Zeile dadurch die Figur an genau zwei Punkten schneidet und der Punkt liegt zwischen ihnen
	Triangle (Dreiecke)	Ein Dreieck, dessen jeder Punkt entweder innerhalb oder auf eine Seite des anderen Dreiecks ist. Aber die Dreiecke nicht übereinstimmen.
Isometric (Isometrisch)	Eine Transformation, bei der Längen erhalten bleiben; nach dem Kongruenzsatz SSS bleiben Winkel auch erhalten.	
Kill (Niederlage)	**Chord (Sehne der Niederlage)**	Eine Strecke von einer Seite eines Winkels zu der anderen

	Circle (Kreis der Niederlage)	Ein Kreis mit dem größten Halbmesser, der zwei Winkel berührt oder innerhalb von denen liegt
Legs (Seiten oder Katheten)	**Triangle (Dreiecks)**	Die Seiten die keine Basis und keine Hypotenuse sind
	Triangle Frustum (Dreieckstumpfes)	Die Seiten sind nicht parallel

Lemma (Hilfsatz oder Lemma) Eine Aussage, die verwendet wird, um andere wichtigere Sätze zu beweisen; im Plural heißt Lemmata

Length (Länge) Das Maß für die Größe eines Segments; der Abstand zwischen seinen Endpunkten

Line (Gerade) Ein Abschnitt, der von beiden Endpunkten verlängert wird; man bezeichnet sie als $\overleftrightarrow{EF}$, wenn $\overline{EF}$ ein Abschnitt ist

Line of Centers (Zentralle) Die Linie, die durch die Mittelpunkte zweier Kreise geht

Lines, Supplementary (Linien, Supplementär) Wenn internen alternativen Winkel Supplementwinkel sind, die zwei Linien, die von der Querlinie geschnitten werden, sind ergänzend nach solcher Querlinie.

Locus (Lokus) Alle Punkte, die eine Bedingung erfüllen; im Plural heißt Lokus

Long (Lang) Innerhalb eines Innenwinkels eines Dreiecks, aber außerhalb des Dreiecks
In der Reichweite eines Maschinengewehrs, aber hinter seiner Sehne der Niederlage

	Center (Mittelpunkt)	Der Schnittpunkt einer Winkelhalbierenden und des Umkreises
	Circle (Kreis)	Um den Mittelpunkt herum durch den Inkreis und Umkreis

Magnitude (Größenordnung) Eine Menge sowohl mit einer Äquivalenzrelation, $=$, als auch einer Totalordnung, $\leq$

Maltitude (Mittelpunkthöhe) Mittelpunkthöhe; das Lot fällt vom Mittelpunkt einer Vierecksseite auf die gegenüberliegende Seite

Measure (Maß) Die Größe der Mengen; Anzahlen diskreter Punkte, Längen der Abschnitte, sowie Flächen von Dreiecken oder von Vereinigungen der disjunkten Dreiecke.

Medial Point (Medialer Punkt) Der Punkt, an dem die Seitenhalbierenden oder die Bimediane zusammenfallen

Median (Seitenhalbierende oder Schwerlinie)	Eine Strecke von einem Eckpunkt eines Dreiecks zum Mittelpunkt der gegenüberliegenden Seite	
Mediator (Halbierende)	Eine Senkrechte, die einen Abschnitt halbiert	
Midpoint (Mittelpunkt)	Der Punkt, an dem eine Figur halbiert wird	
Mirror Property (Spiegeleigenschaft)	Das Licht, das ein Spiegel auf einen Punkt widerspiegelt, hat die gleiche Länge, die es vor seiner Widerspiegelung hätte	
Mid–Segment (Mittelparallele oder Mittellinie)	Triangle (Dreieck)	Eine Strecke, die die Mittelpunkte von zwei Seiten verbindet
	Triangle Frustum (abgeschnittenes Dreieck)	Ein Abschnitt, der die Mittelpunkte der Seiten verbindet
Miquel (Miquel)	Miquel point (Miquel-Punkt)	Der Punkt, der durch den Satz von Miquel definiert ist, wo sich drei Kreise schneiden.
	Miquel circles (Umkreis von Miquel)	Die Umkreise, die durch den Satz von Miquel definiert werden
Napoleon Point (Napoleon-Punkt)	First (Erster)	Das Zentrum des gleichseitigen Dreiecks, das im Napoleon-Satz für gleichseitige äußere Dreiecke definiert wird
	Second (Zweiter)	Das Zentrum des gleichseitigen Dreiecks, das im Napoleon-Satz für gleichseitige innere Dreiecke definiert wird
Neutral Geometry (neutrale Geometrie)	Ein Satz von Postulaten, in dem parallele Geraden nicht erwähnt werden; absolute Geometrie	
Non–Euclidean (Nichteuklidische)	Eine Satzmenge, die einen Satz beinhaltet, der dem Parallelenaxiom widerspricht	
Opposite (gegenüberliegend)	In a Triangle (in einem Dreieck)	Ein Winkel und eine Seite, die einander gegenüberliegen
	Of a Line (von einer Gerade)	Endpunkte eines Abschnitts, der von der Gerade durchgeschnitten wird
	In a Quadrilateral (in einem Viereck)	Zwei Seiten oder zwei Winkel, die einander gegenüberliegen
Ordering (Ordnung)	Eine Menge und Relation, $\leq$, die reflexiv, antisymmetrisch und transitiv ist	
	Total (Gesamtordnung)	$a \leq b$ oder $b \leq a$ für alle Paare a, b in der Menge

Orthic Reflection (Höhenfußpunktreflektion)	Die Reflexion des Höhenschnittpunktes um eine Seite des Dreiecks herum
Orthocenter (Höhenschnittpunkt)	Der Schnittpunkt, an dem alle Höhen des Dreiecks sich überschneiden
Orthogonal (Orthogonal oder Rechtwinklig)	Zwei Bögen, so dass die Tangenten an ihrem Schnittpunkt senkrecht zu einander stehen
Parallel (Parallele)	Zwei Linien, die sich nicht überschneiden
Pedal Point (Fußpunkt)	Ein Punkt, von dem aus den Loten (Senkrechte) auf die Seiten oder die Erweiterungen der Seiten eines Dreiecks oder eines Vierecks fallengelassen werden
Pencil (Geradenbüschel)	Ein Dreieck, dessen Basen innerer und äußerer Winkelhalbierenden, die auf der Seite und der Seitenverlängerung liegen, mit der gegenüberliegenden Spitze verbunden werden (Das ist ein großes Wort in fortgeschrittener Geometrie. Allerdings ist es noch nicht unsere Sache. Nur diese Figur zeichnen, und dann Geradenbüschel nennen reicht für Oberschülern.)
Perimeter (Umfang oder Perimeter)	Die Summe der Längen der Seiten eines Dreiecks oder Vierecks
Perpendicular (Lot)	Eine gerade Linie, deren Schnittpunkt mit einer anderen Linie einen rechten Winkel bildet
Polygon (Vieleck)	Eine Vereinigung mehrerer Dreiecke, die an ihren Seiten so benachbart sind, dass sie konvex ist
Postulate (Postulat)	Die Axiome, die extra für Geometrie sind, nicht aber für andere Zweige der Mathematik
Power of the Point (Potenz eines Punktes)	Für einen Punkt P und einen Kreis mit Mittelpunkt O und Radius r; wenn $z = \lvert\overline{OP}\rvert$, dann ist die Potenz des Punktes $\lvert P\rvert = \lvert r^2 - z^2\rvert$. Wenn P x ist, und y entfernt vom Kreis auf einer Sehne oder einer Sekante liegt, dann $xy = \lvert P\rvert$.
Probability (Wahrscheinlichkeit)	Das Verhältnis die Maß einer Teilmenge zur Maß der Vollmenge
Projection (Projektion)	Die Projektion von $\overline{EG}$ und $\overline{FG}$ auf $\overline{EF}$ im Dreieck $\overline{EFG}$ ist $\overline{EG'}$ bzw. $\overline{FG'}$
Quadrature (Quadratur)	Sätze, die Gleichheit der Flächen von Dreiecken oder Vereinigungen der Dreiecke beweisen

Quadrilateral (Viereck)	Die Vereinigung von zwei Dreiecken, die an einer Seite benachbart sind, so dass sie konvex sind; $\overline{EFGH}$	
	Bi–Centric (Bizentrisches Viereck)	Ein Viereck, das sowohl zyklisch als auch tangential ist
	Contact (Kontaktviereck)	Die Berührungspunkte eines Inkreises eines tangentialen Vierecks, die nacheinander miteinander verbunden sind
	Cyclic (Zyklisches Viereck)	Ein Viereck, das einen Umkreis hat
	Isosceles Kite (Gleichschenkliger Drachen)	Ein Drachenviereck mit zwei gleichlangen Seiten, die gleichlang der Diagonale sind
	Kite (Drachenviereck)	Zwei kongruente Dreiecke in der Abbildung während ihre andere Seite miteinander identisch und auch aufeinanderfolgend sind.
	Lambert (Lambert-Viereck)	Ein Viereck mit drei rechten Winkeln
	Long (Langes)	Das Viereck, dessen Ecken entfernte Zentren sind
	Medial Parallelogram (Mittenparallelogramm)	Die verbundenen Mittelpunkte der benachbarten Seiten eines Vierecks
	Orthodiagonal (Orthodiagonales Viereck)	Ein Viereck, in dem die Diagonalen sich rechtwinklig kreuzen
	Parallelogram (Parallelogramm)	Zwei kongruente Dreiecke in der Abbildung während ihre andere Seite miteinander identisch und auch einander gegenüberliegend sind.
	Parent (Ausgangsviereck)	Das Viereck um ein Mittenparallelogramm herum
	Pedal (Fußpunktviereck)	Die verbundene Fußpunkte, deren Höhen vom Pedalpunkt gezeichnet sind
	Rectangle (Rechteck)	Ein Viereck mit gleichen Winkeln
	Rhombus (Raute)	Ein Viereck mit allen gleichlangen Seiten
	Right Kite (Drachen)	Ein Dreieck, dessen zwei kongruente Dreiecke rechtwinklig sind
	Right Rectangle (Rechteck)	Ein Rechteck mit rechten Winkeln
	Right Square (Quadrat)	Ein Rechteck mit allen gleichlangen Seiten und vier rechten Winkeln
	Saccheri (Saccheri-Viereck)	Ein Viereck mit zwei gleichlangen einander gegenüberliegenden Seiten, die senkrecht zur Basis sind
	Square (Quadratisches)	Ein Rechteck mit gleichen Seiten
	Tangential (Tangentiales)	Ein Viereck, für das ein Inkreis existiert

Radius (Radius oder Halbmesser)	Der Abstand zwischen dem Mittelpunk eines Kreises und der Kreislinie; Radien im Plural	
Random (zufällig)	Die Punkte in einem Abschnitt oder innerhalb eines Dreiecks oder Kreises sind gleichmäßig verteilt	
Ray (Strahl)	Eine Gerade, die in einer Richtung erweitert wird; wird als $\overrightarrow{EF}$ bezeichnet, wenn $\overline{EF}$ ein Teil davon ist	
Reflection (Reflexion)	Punktspiegelung: Zeichnen Sie eine gerade Linie durch einen gegebenen Punkt und einen beliebigen Punkt. Danach verlängern Sie die Linie um die gleiche Länge, die es zwischen zwei Punkte gibt; Punktspiegelung: spiegelt den Punkt in Bezug auf den Lotfußpunkt auf einer Gerade wider; Achsenspiegelung (Geradenspiegelung): spiegelt zwei Punkte in Bezug auf einen Punkt wider und zeichnet eine gerade Linie durch die beiden Punkte; Kreisspiegelung: spiegelt den Mittelpunkt eines gegebenen Kreises in Bezug auf einen Punkt wider und bildet einen gleichen Kreis	
Reflexive Relation (Reflexive Beziehung)	Eine binäre Beziehung über eine Menge, sodass jedes Element mit sich selbst zusammenhängt	
Relation (Relation)	Ein Operator (wahr/falsch) für ein geordnetes Paar von Elementen aus einer gegebenen Menge	
Secant (Sekante)	Eine Gerade, die durch genau zwei Punkte eines Kreises geht	
Segment (Abschnitt)	Alle Punkte auf dem kürzesten Weg zwischen zwei Punkten; $\overline{EF}$	
Semi (Halbe(r))	**Difference (Differenz)**	Die halbe Differenz von zwei Längen oder zwei Winkeln
	Perimeter (Umfang)	Der halbe Umfang eines Dreiecks
	Sum (Summe)	Die halbe Summe von zwei Längen oder zwei Winkeln
Side (Seite)	**Triangle (Dreieck)**	Eines der drei Abschnitte, die ein Dreieck bilden
	Quadrilateral (Viereck)	Eine Figur eines seiner Dreiecke, der keine gemeinsame Seite mit dem anderen Dreieck hat
	Consecutive (Benachbarte)	Die Viereckseiten, die sich eine Ecke teilen
Similar (Ähnlich) $\sim$	Zwei Dreiecke, bei denen alle entsprechenden Winkel gleich sind	
Steiner Line (Steiner-Gerade)	Eine Gerade, die parallel zur Wallace Gerade ist, sodass die Wallace Gerade in der Mitte zwischen der entsprechenden Gerade und dem	

	Pedalpunkt zur Wallace Gerade liegt, und sie ist auch genannt als die Ortholinie oder orthozentrische Linie.	
Subtend (Subtendieren)	Eine Sehne, die eine äquivalente Klasse von eingeschriebenen Winkeln auf einer Seite davon sich bildet.	
Subtend at Center (Subtendieren am Mittelpunkt)	Eine Sehne, die ein Winkel mit seinem Scheitel am Kreismittelpunkt sich bildet	
Summit (Oberseite)	Die Seite eines Saccheri-Vierecks, die der Basis gegenüber liegt	
Symmetric Relation (Symmetrische Relation)	Ein Verhältnis, das für zwei Elemente, wenn sie in beliebiger Reihenfolge betrachtet werden, festgelegt werden kann	
Synthetic (Syntetisch)	Wissen, das nach dem Löschen der Hilfslinien und Bögen verbleibt	
T & V (T & V)	Die Quer- und Vertikalwinkelsätze, die in irgendeiner Kombination verwendet werden	
Tangent (Tangente)	Eine Gerade, die einen Kreis berührt. Wenn eine Länge erwähnt ist, bedeutet dies die Länge des Abschnitts zwischen dem Berührungspunkt und dem Punkt, der die Tangente definiert.	
	Cut (Sekante)	Eine Gerade durch einen Kreis, die zwischen den äußeren Tangenten liegt
	External (Externe)	Eine Gerade, die zwei Kreise tangiert, die nicht zwischen ihren Zentren liegt, oder ein Abschnitt, der zwischen den Berührungspunkten liegt
	Internal (Interne)	Eine Gerade, die zwei disjunkten Kreise tangiert, die zwischen ihren Zentren verläuft, oder ein Teil, der zwischen den Berührungspunkten liegt
Theorem (Theorem oder Satz)	Eine Aussage, die einen Beweis durch Postulate oder andere bereits bewiesene Theoreme erfordert	
Torricelli (Torricelli)	Diese Begriffe gelten nur für Dreiecke mit Winkeln von weniger als 2φ.	
	Apex (Spitze)	Die Spitze eines gleichseitigen Dreiecks, die übe der Außenseite einer Dreiecksseite gezeichnet wird; Wird $\overline{EFG}$ gegeben, E'' liegt E gegenüber.
	2nd Apex (Zweite Spitze)	Die Spitze eines gleichseitigen Dreiecks, die über der Innenseite einer Dreiecksseite gezeichnet wird; in diesem Buch ist sie nicht beschriftet.
	Point (Punkt)	Der Schwerpunkt, an dem kopunktale Torricelli-Segmente sich schneiden; O

	2nd Point (Zweiter Punkt)	Der Schwerpunkt, an dem die zweiten kopunktalen Torricelli-Segmente sich schneiden; V
	Segment (Segment)	Verbinden Sie einen Eckpunkt eines Dreiecks mit dem Torricellispitze; z.B. $\overline{EE''}, \overline{FF''}, \overline{GG''}$
	2nd Segment (Zweites Segment)	Verbinden Sie einen Eckpunkt eines Dreiecks mit der zweiten Torricelli-Spitze
	(In der Literatur werden diese Dinge oft auch nach Pierre de Fermat benannt.)	
Touch (Tangieren)	Eine Gerade und ein Kreis oder zwei Kreise überschneiden miteinander in nur einem Punkt, so dass sie nicht miteinander durchschneiden	
	Touching Point (Berührungspunkt)	Der Punkt, an dem sich eine Gerade und ein Kreis oder zwei Kreise berühren
Transformation (Transformation)	Die Beziehung zwischen zwei Punktemengen, die eins zu eins und surjektiv ist; das heißt, genau ein Punkt der einen Menge ist jedem Punkt anderer Menge zugeordnet.	
Transitive Relation (Transitive Relation)	Wenn eine Relation wahr für a und b ist und für b und c, dann ist sie wahr für a und c	
Transversal (Querlaufend)	Eine Linie, die zu keiner von zwei gegebenen Linien parallel ist	
Traverse (Traverse)	Seitliche Drehung eines Maschinengewehrs auf seinem Stativ	
	Top (Winkel des horizontalen Feuers)	Der maximale Winkel, den eine Maschinengewehr durchqueren kann
Triangle (Dreieck)	Die Abschnitte, die drei nicht kollineare Punkte verbinden, sogenannte Eckpunkte; z.B. $\overline{EFG}$	
	Acute (Spitzwinkliges)	Ein Dreieck, bei dem alle Winkel spitz sind
	Antipedal (Antifußpunktdreieck)	Ein Dreieck, für das ein angegebenes Dreieck relativ zu einem angegebenen Fußpunkt ein Fußpunktdreieck ist
	Contact (Kontaktdreieck)	Ein Fußpunktdreieck, wenn ein Inkreismittelpunkt gleichzeitig ein Fußpunkt ist
	Crossed (Gekreuztes)	Zwei Sekanten, die sich innerhalb zweier paralleler Linien schneiden; lassen Sie die Enden weg, um Dreiecke zu sehen
	Degenerate (Entartetes)	Die Eckpunkte sind kollinear; es ist kein Dreieck

	Double–Long (Doppel-Langes)	Das Dreieck, dessen Ecken exzentrisch sind; exzentrisch
	Egyptian (3-4-5-Dreieck)	Ein Dreieck mit den Seiten 3e, 4e und 5e, wobei e eine beliebige Einheitsstrecke ist
	Equilateral (Gleichseitiges)	Ein Dreieck, bei dem alle drei Seiten gleich lang sind
	Half Equilateral (Halbes Gleichseitiges)	Ein gleichseitiges Dreieck, das durch seine Höhe halbiert wird
	Isosceles (Gleichschenkliges)	Ein Dreieck, bei dem zwei Seiten gleichlang sind
	Long (Langes)	Das Dreieck, dessen Ecken entfernte Zentren sind
	Medial (Mittendreieck)	Drei Mittelparallelen eines Dreiecks bilden seine Seiten
	Nested (Verschachteltes)	Zwei Sekanten, die sich außerhalb zweier paralleler Linien schneiden; lassen Sie die Enden weg, um das Dreieck zu sehen
	Obtuse (Stumpfes)	Ein Dreieck mit einem stumpfen Winkel
	Orthic (Höhenfußpunktdreieck)	Verbinden Sie die Fußpunkte der drei Höhen eines Dreiecks
	Parent (Ausgangsdreieck)	Ein Dreieck, aus dem ein Mittendreieck stammt
	Pedal (Fußpunktdreieck)	Die verbundene Fußpunkte, deren Höhen vom Pedalpunkt gezeichnet sind
	Right (Rechtwinkliges)	Ein Dreieck mit einem rechten Winkel
	Scalene (Ungleichseitiges)	Ein Dreieck mit unterschiedlichen Seitenlangen
	Tangential (Tangentendreieck)	Seine Seiten sind tangierend zum Umkreis auf den Eckpunkte von ¯EFG
	Too Obtuse (zu stumpf)	Ein Dreieck mit einem Winkel, der gleich oder größer als 2φ ist
	Viviani (Viviani)	$\overline{EPP_F}$ und $\overline{FPP_E}$, wenn $\overline{EFG}$ gleichschenkliges Dreieck mit Basis $\overline{EF}$ und einem Punkt P irgendwo dazwischen E und F ist
Trichotomy (Trichotomie)	Beweis durch Widerspruch, wenn es drei Ausweichmöglichkeiten gibt	
Tri–Segment (Tri-Abschnitt)	**Triangle (Dreieck)**	Ein Abschnitt, der die Trisektion-Punkte entweder in der Nähe der Basis oder in der Nähe der Spitze verbindet
Undefined Terms (undefinierte Begriffe)	Zu den intuitiven Konzepten gehören Ebene, Punkt, kürzester Weg, Gerade	

Under Defined (Unterspezifiziert)	Keine ausreichenden Informationen; die Anzahl der möglichen Lösungen ist unendlich
Vertex (Scheitel oder Spitze)	Der Schnittpunkt zweier Linien, Strahlen oder Seiten eines Dreiecks oder Vierecks
Visible Under an Angle (Sichtbar unter dem Winkel)	Alle Punkte innerhalb des Winkels; Feuerfeld
Wallace-Line (Wallacesche Gerade)	Ein Höhenfußpunktdreieck entartet zu einer Geraden, wenn sein Höhenschnittpunkt auf dem Umkreis liegt (In der Literatur wird es oft Simson-Gerade genannt.)
Width (Breite)	Die Länge der Senkrechte zwischen zwei Geraden **Shoulder (einer Achsel)** Die Höhe eines Dreiecks eines Vierecks, das durch seine Diagonale halbiert wird

Англо–русский глоссарий

Adjacent (примыкающие или прилежащие)	Два неперекрывающихся треугольника с общей стороной (общей по всей длине) или два угла с общим лучом (общая вершина и направление)
Altitude (высота)	Перпендикуляр, опущенный из вершины треугольника на продолжение противолежащей стороны
Analytic (аналитический)	Знания, содержащиеся в данной информации
Angle (угол)	Два луча, называемые сторонами, имеющие общий конец, называемый вершиной. $\angle F$ если речь идет об одном угле в F, или $\angle EFG$ для угла между $\overrightarrow{FE}$, $\overrightarrow{FG}$

Acute (острый)	Угол, который меньше прямого угла
Alternate Interior (внутренние накрест лежащие)	Углы на противоположных сторонах секущей и между двумя данными прямыми
Apex (при апексе)	Угол, лежащий напротив основания треугольника
Base (при основании)	В треугольнике, имеющем основание, углы на двух концах основания
Central (центральный)	Угол, вершина которого является центром окружности
Complementary (дополнительные)	Два угла, которые в сумме дают один прямой угол
Conjugate (сопряженные)	Углы, которые в сумме дают два развернутых угла
Consecutive (односторонние)	Два угла на концах стороны, оба из которых являются внутренними (внешними)
Elevation (возвышения)	Один луч лежит в горизонтальной плоскости, а второй выше нее
Exterior (внешний)	Угол, смежный внутреннему углу
Inscribed (вписанный)	Угол внутри окружности, вершина которого лежит на этой окружности
Interior (внутренний)	Угол внутри треугольника или четырехугольника при вершине
Obtuse (тупой)	Угол, который больше прямого угла и меньше развёрнутого
Parallelism (параллельности)	В гиперболической геометрии; $2\,\mathrm{atan}(e^{-x})$, где x высота
Right (прямой)	Разделенный пополам развернутый угол
Skew (асимметрии)	Разность углов при основании треугольника

	Straight (развёрнутый)	Угол, лучи которого лежат на одной прямой и расходятся в противоположные стороны
	Supplementary (дополняющие до развёрнутого)	Два угла, которые в сумме дают один развернутый угол
	Vertical (вертикальные)	Углы, лежащие напротив друг друга при пересечении двух прямых

Anticenter (антицентр)	Точка, в которой сходятся серединные высоты; существует для циклических четырехугольников
Apex (апекс)	Вершина треугольника, противолежащая основанию
Arc (дуга)	Часть окружности; в равных окружностях центральные углы и дуги, на которые они опираются, являются преобразованием друг друга
Area (площадь)	Мера, характеризующая размер треугольника или объединения неперекрывающихся треугольников
Auxiliary (вспомогательный)	Не заданные прямые или дуги, пересечение которых выходит за пределы аналитики
Axiom (аксиома)	Утверждение, которое принимают без доказательства ради изучения его следствий
Base (основание)	Сторона равнобедренного треугольника, заключенная между равными углами Сторона треугольника, которую обозначили таким образом, или та сторона, на которой треугольник построен
Between (между)	1. Если F находится между E и G, то F также находится между G и E, и существует прямая, содержащая точки E, F, G ("Между" подразумевает, что данные три точки различны.); 2. Если E и G две точки на прямой, то существует по меньшей мере одна точка F, лежащая между E и G, и по меньшей мере одна точка H такая, что G лежит между E и H; 3. Среди любых трех точек, лежащих на одной прямой, ровно одна находится между двумя другими
Bi–Conditional (биусловие)	Высказывание вида p тогда и только тогда, когда q. Оно истинно, если p и q оба истинны или p и q оба ложны. Из p следует q; также из q следует p. В доказательстве одной импликации нельзя использовать другую, и наоборот. Тогда и только тогда сокращенно обозначается ттт.
Bi–Medial (бимедиаль)	Пересечение диагоналей четырехугольника
Bimedian (бимедиана)	Отрезок, соединяющий средние точки противолежащих сторон четырехугольника

Bisect (делить пополам) $\frac{1}{2}$	Делить отрезок или угол на две равные части, называемые половинами
Center Line (центровая линия)	Медиатриса основания равнобедренного треугольника или полуокружности
Centroid (центроид)	Точка равновесия однородной пластины. Доказательство того, что серединная точка треугольника является его центроидом, требует вычислений, поэтому данные термины *не* являются взаимозаменяемыми.
Chord (хорда)	Отрезок между двумя точками на окружности
	Common (общая) — Отрезок между точками пересечения двух окружностей
Circle (окружность)	Все точки, равноудаленные от данной точки, которая называется центром
Circum (описанная)	**Circle (окружность)** — Окружность, которая пересекает фигуру в ее вершинах
	Center (центр) — Центр описанной окружности
	Radius (радиус) — Радиус описанной окружности; R
Closing the Horizon (Закрывающий горизонт)	Все углы вокруг точки должны в сумме давать 2σ
Collinear (лежащие на одной прямой)	Множество точек, все из которых лежат на одной прямой
Concentric (концентрические)	Две или более окружности с одинаковым центром, но разными радиусами
Concurrent (иметь общую точку пересечения)	Три или более прямые или дуги, которые пересекаются в одной точке
Concyclic (лежащие на одной окружности)	Четыре или более точки на одной окружности
Condition (условие)	Ограничения, которым фигура либо соответствует, либо нет
Conformal (конформный)	Преобразование, сохраняющее углы; например, масштабирование и инверсия относительно окружности
Congruent (конгруэнтный) $\cong$	Два треугольника, у которых площади, стороны и внутренние углы равны

Contradiction, Proof by (доказательство от противного)	Чтобы доказать, что из высказывания p следует высказывание q, предполагаем, что p истинно, а q не истинно, и показываем, что это невозможно
Converse (обратный)	Если дано высказывание, что из p следует q, высказывание, что из q следует p
Convex (выпуклый)	Любой отрезок, лежащий между двумя точками, которые являются внутренними по отношению к двум сторонам, находится внутри фигуры
Defect (дефект)	В гиперболической геометрии; $\sigma - (\alpha + \beta + \gamma)$ для треугольника данного размера
Diagonal (диагональ)	Отрезки, соединяющие не прилежащие к одной стороне вершины четырехугольника **Definitional (определяющая)** — Общая сторона двух треугольников в четырехугольнике
Diameter (диаметр)	Хорда, которая проходит через центр окружности **Diametrically Opposed (диаметрально противоположные)** — Концы диаметра
Dichotomy (дихотомия)	Доказательство от противного, когда имеется две альтернативы
Discussion (обсуждение)	Необходимые и достаточные условия решения и сколько решений
Disjoint (неперекрывающиеся)	Фигуры, которые не накладываются друг на друга; их площади образуют аддитивную группу (Сюда входят касающиеся окружности и примыкающие треугольники)
Disjoint (неперекрывающиеся)	Вероятность того, что какая-либо точка лежит внутри обеих фигур, равна нулю
Edubabble (педтрёп)	Напыщенное пустословие и заумные фразы, которыми стремятся запутать и пустить пыль в глаза. (Американские учителя старших классов не изучают свой предмет в университете – эти знания они черпают из учебника своих учеников – вместо этого их пичкают педагогическим трепом)
Endpoint (конец)	Точка в конце отрезка, дуги или луча
Equal (равные) $=$	Сравнимые величины, которые не меньше и не больше друг друга
Equidistant (равноудаленные)	Две пары точек, которые определяют два отрезка равной длины

	Lines (прямые)	Любые два перпендикуляра между ними имеют равную длину
Equivalence (эквивалентность)	Class (класс эквивалентности)	Множество объектов, которые равны, конгруэнтны, подобны или параллельны
	Relation (отношение эквивалентности)	Множество и рефлексивное, симметричное и транзитивное отношение
Equivalent (эквивалентные)	Условия, любые два из которых представляют собой биусловие	
Euler (Эйлер)	Center (центр)	Центр окружности Эйлера
	Circle (окружность Эйлера)	В треугольнике: окружность, проходящая через средние точки сторон, основания высот и средние точки отрезков, соединяющих ортоцентр с вершинами. В некоторых учебниках её также называют окружностью девяти точек.
	Segment (отрезок Эйлера)	Отрезок, соединяющий ортоцентр с центром описанной окружности
Ex (вневписанная)	Circle (окружность)	Окружность, касающаяся одной стороны треугольника и продолжений прилежащих к ней сторон
	Center (эксцентр)	Центр вневписанной окружности
	Foot (основание внешней биссектрисы)	Место, где биссектриса внешнего угла пересекает продолжение противолежащей стороны; $E^{\times}, F^{\times}, G^{\times}$
	Radii (радиусы)	Радиусы вневписанных окружностей; для $\omega_X, \omega_Y, \omega_Z$ это r_X, r_Y, r_Z
Extend (продолжить, продлить)	Если дано $\overline{EF}$, построить $\overline{EG}$, такой что F внутри $\overline{EG}$ или E внутри $\overline{FG}$	
Field of Fire (зона обстрела)	Все точки, которые являются внутренними по отношению к максимальному поперечному повороту пулемёта	
Figure (фигура)	Множество точек. Они могут быть отдельными или соединяться в прямые, отрезки или дуги	
Foot (основание)	Пересечение, когда из точки на прямую опускают перпендикуляр	

Frustum, Triangle (усечённый треугольник)	Часть треугольника между основанием и сечением, параллельным основанию	
Fully Defined (однозначно определенный)	Фигура, обладающая данными свойствами, существует и единственна	
Geometric Mean (среднее геометрическое)	Если $\frac{a}{b} = \frac{b}{c}$ верно для вещественных чисел a, b, c, то b является средним геометрическим a и c	
Half–Scale (половинный)	Треугольник, стороны которого составляют половину соответствующих сторон другого треугольника	
Harmonic Division (гармоническое деление)	Разделить отрезок $\overline{EF}$ на две части таким образом, чтобы меньшая часть относилась к большей, как большая ко всему отрезку $\overline{EF}$; $\frac{\overline{EG^*}}{\overline{FG^*}} = \frac{\overline{EG^\times}}{\overline{FG^\times}}$	
Homothetic (гомотетичный)	Center (центр гомотетии)	$\overleftrightarrow{E''E} \cap \overleftrightarrow{F''F} \cap \overleftrightarrow{G''G}$ является центром гомотетии, которой для $\overline{EFG}$ получают гомотетичное удвоение $\overline{E''F''G''}$
	Double (гомотетичное удвоение)	Треугольник, у которого стороны вдвое длиннее сторон другого треугольника, а продолжения сторон попарно параллельны продолжениям сторон этого треугольника
	Triple (гомотетичное утроение)	Аналогично гомотетичному удвоению, но с тройным увеличением длины
Hypotenuse (гипотенуза)	Сторона прямоугольного треугольника напротив прямого угла	
In (вписанная)	Circle (окружность)	Окружность, которая касается всех сторон фигуры
	Center (инцентр)	Центр вписанной окружности
	Diameter (диаметр)	Диаметр вписанной окружности; d
	Foot (основание внутренней биссектрисы)	Место, где биссектриса угла пересекает противолежащую сторону; E^*, F^*, G^*
	Radius (радиус вписанной окружности)	Радиус вписанной окружности; r
Inside (внутри)	Segment (отрезок)	Член множества точек отрезка, но не конец отрезка

	Figure (фигура)	Такая точка, что любая проведенная через неё прямая пересекает фигуру ровно в двух точках, причём данная точка лежит между ними
	Triangle (треугольники)	Треугольник, каждая точка которого лежит либо внутри другого треугольника, либо на одной из его сторон, но треугольники не совпадают
Isometric (изометрический)	Преобразование, которое сохраняет длины; согласно теореме о конгруэнтности треугольников по трем сторонам, оно также сохраняет углы	
Kill (поражения)	Chord (хорда поражения)	Отрезок, проведенный от одной стороны угла к другой
	Circle (окружность поражения)	Окружность наибольшего радиуса, которая касается или находится внутри двух углов
Legs (катеты или боковые стороны)	Triangle (в треугольнике)	Стороны, отличные от основания или гипотенузы
	Frustum, Triangle (в усечённом треугольнике)	Те стороны, которые не параллельны
Lemma (лемма)	Теорема, используемая для доказательства других, более важных теорем	
Length (длина)	Мера, характеризующая размер отрезка; расстояние между его концами	
Line (прямая)	Отрезок, продленный с обоих концов; обозначается $\overleftrightarrow{EF}$, если $\overline{EF}$ отрезок	
Line of Centers (линия центров)	Прямая, которая проходит через центры двух окружностей	
Lines, Supplementary (дополняющие прямые)	Если при пересечении двух прямых секущей внутренние накрест лежащие углы дополняют друг друга до развернутого, то данные две прямые являются дополняющими относительно этой секущей	
Locus (геометрическое место точек)	Все точки, которые удовлетворяют условию	
Long (отдалённый/ в отдалении от)	Внутри внутреннего угла треугольника, но за пределами треугольника	
	Внутри зоны обстрела пулемета, но вне его хорды поражения	
	Center (центр)	Пересечение биссектрисы угла и описанной окружности

	Circle (окружность)	Вокруг отдалённого центра через центр вписанной окружности и центр вневписанной окружности
Magnitude (величина)	Множество, на котором одновременно задано отношение эквивалентности, $=$, и отношение линейного порядка, $\leq$	
Maltitude (антимедиатриса)	Высота из средней точки; перпендикуляр, опущенный из средней точки стороны четырёхугольника на противолежащую сторону	
Measure (мера)	Размер множеств; подсчёты дискретных точек, длины отрезков, площади треугольников или объединений неперекрывающихся треугольников	
Medial Point (медиальная точка)	Точка, в которой сходятся медианы или бимедианы	
Median (медиана)	Отрезок, проведенный от вершины треугольника к средней точке противолежащей стороны	
Mediator (медиатриса)	Серединный перпендикуляр отрезка	
Midpoint (средняя точка)	Точка, которой отрезок делится пополам	
Mirror Property (зеркальное свойство)	Свет, отскакивающий от зеркала в некоторую точку, проходит такое же расстояние, какое прошел бы до ее отражения	
Mid–Segment (средняя линия)	**Triangle (треугольника)**	Отрезок, соединяющий средние точки двух сторон
	Triangle Frustum (усечённого треугольника)	Отрезок, соединяющий средние точки боковых сторон
Miquel (Микель)	**Miquel point (точка Микеля)**	Точка, определенная теоремой Микеля, где сходятся три окружности
	Miquel circles (окружности Микеля)	Описанные окружности, определенные теоремой Микеля
Napoleon Point (точка Наполеона)	**First (первая)**	Центр равностороннего треугольника, определенного теоремой Наполеона для внешних равносторонних треугольников
	Second (вторая)	Центр равностороннего треугольника, определенного теоремой Наполеона для внутренних равносторонних треугольников

Neutral Geometry (нейтральная геометрия)	Набор постулатов, в котором не упоминаются параллельные прямые; абсолютной геометрии	
Non–Euclidean (неевклидов)	Набор постулатов, содержащий один постулат, который противоречит постулату о параллельности	
Opposite (противоположные, противолежащие)	In a Triangle (в треугольнике)	Угол и сторона лежащие напротив друг друга
	Of a Line (о прямой)	Концы отрезка, пересекаемого данной прямой
	In a Quadrilateral (в четырёхугольнике)	Две стороны или два угла, лежащие напротив друг друга
Ordering (отношение порядка)	Множество и отношение, $\leq$, которое является рефлексивным, анти-симметричным и транзитивным	
	Total (отношение линейного порядка)	$a \leq b$ или $b \leq a$ для любых a, b множества
Orthic Reflection (ортоцентрическое отражение)	Отражение ортоцентра относительно стороны треугольника	
Orthocenter (ортоцентр)	Точка, в которой сходятся высоты треугольника	
Orthogonal (ортогональные)	Две дуги, такие что касательные к их пересечению перпендикулярны	
Parallel (параллельные)	Две прямые, которые не пересекаются	
Pedal Point (подерная точка)	Точка, из которой опущены перпендикуляры на стороны или продолжения сторон либо треугольника, либо четырехугольника	
Pencil (пучок)	Треугольник, у которого основания внутренней и внешней биссектрис, лежащие, соответственно, на стороне и продолжении этой стороны, соединены с противолежащей вершиной. (На острие геометрической науки пучкам уделяют много внимания, но мы пока до этого уровня не дошли. Для средней школы достаточно просто начертить эту фигуру и назвать ее пучком)	
Perimeter (периметр)	Сумма длин сторон треугольника или четырехугольника	
Perpendicular (перпендикуляр)	Прямая, которая при пересечении с другой прямой образует прямой угол	
Polygon (многоугольник)	Объединение нескольких примыкающих друг к другу треугольников, такое что результат объединения является выпуклым	

Postulate (постулат)	Аксиомы, характерные для геометрии, в отличие от других отраслей математики
Power of the Point (степень точки)	Для точки P и окружности с центром O и радиусом r; если $z = \lvert\overline{OP}\rvert$, то степень точки $\lvert P\rvert = \lvert r^2 - z^2\rvert$. Если P — это x, а y лежит на расстоянии от окружности на хорде или секущей, то $xy = \lvert P\rvert$
Probability (Вероятность)	Отношение меры подмножества к мере всего множества
Projection (проекция)	В треугольнике $\overline{EFG}$ проекцией $\overline{EG}$ и $\overline{FG}$ на $\overline{EF}$ является $\overline{EG'}$ и $\overline{FG'}$, соответственно
Quadrature (квадратура)	Теоремы, доказывающие равенство площадей треугольников или объединений треугольников
Quadrilateral (четырехугольник)	Объединение двух примыкающих друг к другу треугольников, такое что результат объединения является выпуклым; $\overline{EFGH}$

Bi–Centric (бицентричный)	Четырехугольник, который одновременно является циклическим и тангенциальным
Contact (контактный)	Точки касания вписанной окружности тангенциального четырехугольника, последовательно соединенные друг с другом
Cyclic (циклический)	Четырехугольник, для которого существует описанная окружность
Isosceles Kite (равнобедренный дельтоид)	Дельтоид, у которого одна пара равных сторон равна диагонали
Kite (дельтоид)	Объединение двух конгруэнтных треугольников; равные необщие стороны также являются смежными.
Lambert (Ламберта)	Четырехугольник с тремя прямыми углами
Long (отдалённый)	Четырехугольник, вершины которого являются отдаленными центрами
Medial Parallelogram (серединный параллелограмм)	Соединённые средние точки смежных сторон четырёхугольника
Orthodiagonal (ортодиагональный)	Четырехугольник, диагонали которого перпендикулярны
Parallelogram (параллелограмм)	Объединение двух конгруэнтных треугольников; равные необщие стороны являются противолежащими.

Parent (исходный)	Четырехугольник, построенный вокруг серединного параллелограмма
Pedal (подерный)	Соединенные основания перпендикуляров, опущенных из подерной точки
Rectangle (прямоугольник)	Четырехугольник с равными углами
Rhombus (ромб)	Четырехугольник, у которого все стороны равны
Right Kite (прямоугольный дельтоид)	Дельтоид, два конгруэнтных треугольника которого являются прямоугольными
Right Rectangle (прямой прямоугольник)	Прямоугольник с прямыми углами
Right Square (прямой квадрат)	Прямой прямоугольник с равными сторонами
Saccheri (Саккери)	Четырехугольник, у которого две противолежащие стороны равны и перпендикулярны основанию
Square (квадрат)	Прямоугольник с равными сторонами
Tangential (тангенциальный)	Четырехугольник, для которого существует вписанная окружность

Radius (радиус) Отрезок, который проходит от центра окружности к окружности

Random (случайный) Точки на отрезке или внутри треугольника или окружности распределены равномерно

Ray (луч) Отрезок, продленный в одном направлении; обозначается $\overrightarrow{EF}$, если $\overline{EF}$ отрезок

Reflection (отражение) Отражение точки: проводим прямую через данную точку и произвольную точку и затем продолжаем эту прямую на расстояние, равное расстоянию между двумя точками;
Отражение точки: отражаем точку относительно основания перпендикуляра, опущенного на прямую;
Отражение прямой: отражаем две точки относительно некоторой точки и проводим через них прямую;
Отражение окружности: отражаем центр данной окружности относительно точки и строим равную окружность

Reflexive Relation (рефлексивное отношение) Такое бинарное отношение на множестве, в котором всякий элемент находится в отношении с собой

Relation (отношение) Истинностный оператор на упорядоченной паре элементов данного множества

Secant (секущая)	Прямая, которая пересекает окружность ровно в двух точках	
Segment (отрезок)	Все точки вдоль кратчайшего пути между двумя точками; $\overline{EF}$	
Semi (полу-)	Difference (разность)	Половина разности двух длин или двух углов
	Perimeter (периметр)	Половина периметра треугольника
	Sum (сумма)	Половина суммы двух длин или двух углов
Side (сторона)	Triangle (треугольника)	Один из трех отрезков, образующих треугольник
	Quadrilateral (четырехугольника)	Необщий отрезок одного из его треугольников
	Consecutive (смежные)	Стороны четырехугольника, у которых общий конец
Similar (подобные) ~	Два треугольника, у которых все соответствующие углы равны	
Steiner Line (прямая Штейнера)	Прямая, параллельная прямой Уоллеса, такая что прямая Уоллеса проходит посередине между ней и своей подерной точкой; её также называют ортопрямой или ортоцентрической прямой	
Subtend (стягивать)	Хорда, создающая класс эквивалентности вписанных углов по одну сторону от себя	
Subtend at Center (стягивать центральный угол)	Хорда, создающая угол с вершиной в центре окружности	
Summit (верхняя сторона)	Сторона четырёхугольника Саккери, противоположная основанию	
Symmetric Relation (симметричное отношение)	Отношение, которое может быть установлено применительно к двум элементам в любом порядке	
Synthetic (синтетический)	Знания, которые остаются после того, как мы стираем вспомогательные прямые и дуги	
T & V (СВ)	Теоремы о секущей и о вертикальных углах, применяемые в том или ином сочетании	
Tangent (касательная)	Прямая, которая касается окружности; если речь идёт о длине, имеется в виду длина отрезка между точкой касания и точкой, которая определяет касательную	
	Cut (усеченная)	Отрезок внутренней касательной, лежащий между внешними касательными

	External (внешняя)	Прямая, касательная к двум окружностям, которая не проходит между их центрами, либо отрезок, лежащий между точками касания
	Internal (внутренняя)	Прямая, касательная к двум неперекрывающимся окружностям, которая проходит между их центрами, либо отрезок, лежащий между точками касания
Theorem (теорема)	Высказывание, требующее доказательства с использованием постулатов или уже доказанных теорем	
Torricelli (Торричелли)	Эти термины относятся только к треугольникам с углами меньше 2φ	
	Apex (апекс Торричелли)	Апекс равностороннего треугольника, построенного снаружи стороны треугольника; если дано $\overline{EFG}$, E'' лежит напротив E
	2nd Apex (2-й апекс Торричелли)	Апекс равностороннего треугольника, построенного на стороне треугольника внутри последнего; в нашей книге не имеет обозначений
	Point (точка Торричелли)	Точка, в которой сходятся отрезки Торричелли; O
	2nd Point (2-я точка Торричелли)	Точка, в которой сходятся 2-е отрезки Торричелли; V
	Segment (отрезок Торричелли)	Соединяем вершину треугольника с апексом Торричелли; например $\overline{EE''}, \overline{FF''}, \overline{GG''}$
	2nd Segment (2-й отрезок Торричелли)	Соединяем вершину треугольника со 2-м апексом Торричелли
	(В литературе эти понятия часто называют именем Пьера Ферма)	
Touch (касаться)	Прямая и окружность или две окружности касаются, когда они пересекают друг друга ровно в одной точке; то есть, они не заходят друг за друга	
	Touching Point (точка касания)	Точка, в которой касаются прямая и окружность или две окружности
Transformation (преобразование)	Отношение между двумя множествами точек, которое является взаимно однозначным и сюръективным; то есть, каждой точке одного множества ставится в соответствие ровно одна точка другого множества	
Transitive Relation (транзитивное отношение)	Если отношение истинно для a и b и для b и c, то оно истинно для a и c	

Transversal (секущая)	Прямая, которая не параллельна ни одной из двух данных прямых	
Traverse (горизонтальный поворот)	Поперечное вращение пулемета на треноге	
	Top (угол горизонтального обстрела)	Максимальный угол, на который пулемет может повернуться в поперечном направлении
Triangle (треугольник)	Отрезки, соединяющие три не лежащие на одной прямой точки, называемые вершинами; например $\overline{EFG}$	
	Acute (остроугольный)	Треугольник, у которого все углы острые
	Antipedal (антиподерный)	Треугольник, относительно которого данный треугольник является подерным для данной подерной точки
	Contact (контактный)	Подерный треугольник, если центр вписанной окружности является подерной точкой
	Crossed (накрест лежащие)	Две секущие, которые пересекаются между двумя параллельными прямыми; отсекаем хвосты, чтобы увидеть треугольники
	Degenerate (вырожденный)	Вершины лежат на одной прямой; это не треугольник
	Double–Long (дважды отдаленный)	Треугольник, вершины которого являются центрами вневписанных окружностей; эксцентральный
	Egyptian (египетский)	Треугольник, длины сторон которого равны 3, 4 и 5 единицам
	Equilateral (равносторонний)	Треугольник, у которого все стороны равны
	Half Equilateral (полу-равносторонний)	Равносторонний треугольник, рассеченный по своей центральной линии
	Isosceles (равнобедренный)	Треугольник с двумя равными сторонами
	Long (отдалённый)	Треугольник, вершины которого являются отдаленными центрами
	Medial (серединный)	Три средних линии треугольника в качестве его сторон
	Nested (соответственные)	Две секущие, которые пересекаются вне двух параллельных прямых; отсекаем хвосты, чтобы увидеть треугольники
	Obtuse (тупоугольный)	Треугольник с одним тупым углом

Orthic (ортотреугольник) Соединяем основания высот треугольника

Parent (исходный) Треугольник, из которого получают серединный треугольник

Pedal (подерный) Соединенные основания перпендикуляров, опущенных из подерной точки

Right (прямоугольный) Треугольник с одним прямым углом

Scalene (разносторонний) Треугольник, у которого все стороны не равны

Tangential (тангенциальный) Стороны касаются описанной окружности в вершинах $\overline{EFG}$

Too Obtuse (слишком тупоугольный) Треугольник, у которого один угол больше или равен 2φ

Viviani (Вивиани) $\overline{EPP_F}$ и $\overline{FPP_E}$, если $\overline{EFG}$ равнобедренный треугольник с основанием $\overline{EF}$ и подерной точкой P где угодно между E и F

Trichotomy (трихотомия) Доказательство от противного, когда имеется три альтернативы

Tri–Segment (три-сегмент)

Triangle (треугольника) Отрезок, соединяющий точки трисекции, либо обе вблизи основания, либо обе вблизи апекса

Undefined Terms (неопределённые термины) Интуитивные понятия: плоскость, точка, кратчайший путь, развернутый

Under Defined (неоднозначно определённый) Дано недостаточно информации; имеется бесконечное количество решений

Vertex (вершина) Пересечение двух прямых, лучей, или сторон треугольника или четырехугольника

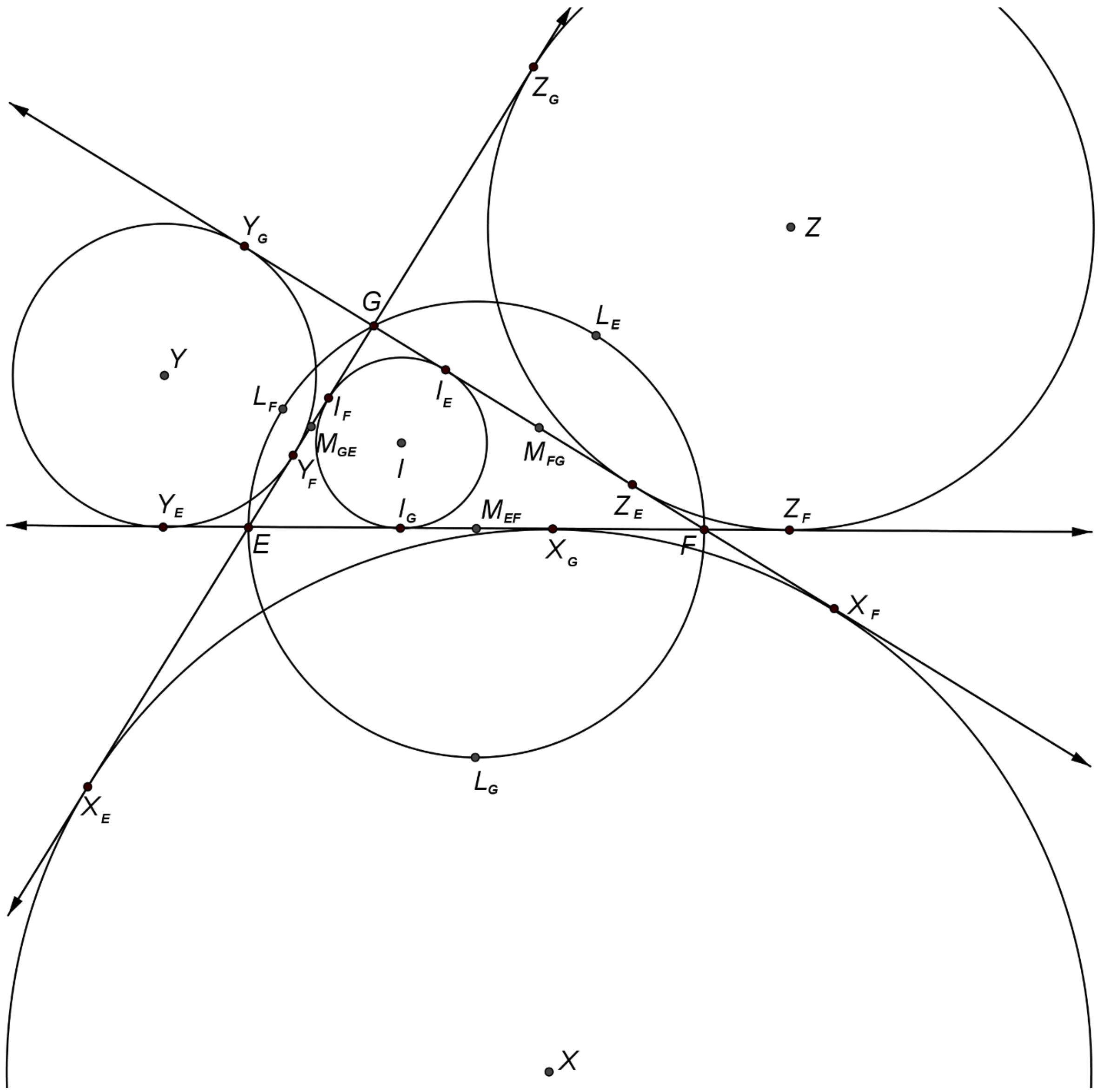
Visible Under an Angle (виден под углом) Все точки внутри угла; зона обстрела

Wallace Line (прямая Уоллеса) Подерный треугольник представляет собой прямую, если подерная точка лежит на описанной окружности
(В литературе это часто называют прямой Симсона)

Width (ширина) Если даны две параллельные прямые, длина перпендикуляра между ними

Shoulder (плеча) Высота определяющего треугольника прямоугольника

STUDENT NOTES



Most textbook authors just randomly pick a bunch of letters! Indeed, organization is my principal contribution to geometry; there is no Aguilar theorem, though my friend Professor Zlatanović did contribute original work. I have collected results proven by geometers all over the world since [1868] 1899, when Wentworth published *Plane Geometry*, the last serious American high school geometry textbook. In [1935] 1958, Wolfe and Phelps published *Practical Shop Mathematics*, which has a chapter on geometry with "fifty propositions… of much greater value to the high school student who is not going to college than is the usual geometry course consisting of about one hundred and fifty theorems (p. v)." Being more readily available than *Plane Geometry* (until it was reprinted in 2007, as was Altshiller-Court and Roger Johnson), many aspiring engineers read Wolfe and Phelps at home when their high school abandoned geometry by turning it into remedial algebra. *Geometry–Do* is like bringing Wentworth back, but with over 300 theorems, theorem names instead of numerical citations, mathlete training, and absolute geometry.